TITLE Metrized Laminations and Quasisymmetric Maps

AUTHOR Oliver A. Goodman

INSTITUTION and DATE University of Warwick, Coventry. CV4 7AL June 1989

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Oliver A. Goodman

Submitted for the degree of PhD.
Department of Mathematics,
University of Warwick,
Coventry. CV4 7AL
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Abstract

Teichmüller space is defined as a space of hyperbolic structures on a surface rather than as a space of conformal structures. Earthquakes are defined and we see how they correspond to hyperbolic structures, via homeomorphisms of the circle. Metrized laminations are defined and we obtain a correspondence with earthquakes. We deduce a correspondence between measured laminations and earthquakes. We define uniform boundedness of earthquakes and show that such earthquakes are surjective. Quasisymmetric maps are defined and investigated. We show that an earthquake is uniformly bounded if and only if its boundary mapping is quasisymmetric. Finally we show how a uniformly bounded earthquake can be approximated, in a natural fashion, by a bi-Lipschitz diffeomorphism.

Metrized Laminations and Quasisymmetric Maps

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1 Introduction

In this thesis we show how Bers' analytic approach to the study of Teichmüller spaces may be linked with Thurston's more geometrical viewpoint. The scope of this paper is described in Section 1.2. Most of the results we obtain are stated but not proved by Thurston in [6]. I would like to thank Professor David Epstein for suggesting the subject matter of this paper and for the time he has put into commenting on my work.

As this is intended to be a self contained exposition we will begin with a few definitions. These are equivalent to definitions given in Thurston [6].

1.1 Definitions, notation and background

Let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$. Let d denote the Poincaré metric on $\mathbb{D}$. Let g be any other complete hyperbolic metric on $\mathbb{D}$. We call g a *relative hyperbolic metric on* $(\mathbb{D}, d)$ if there exists a homeomorphism h of $\mathbb{D}$ such that $h|_{\mathbb{D}}$ is an isometry from g to d. We refer to d as the *reference metric*.

Let g_0 and g_1 be relative hyperbolic metrics on $(\mathbb{D}, d)$. We say that g_0 is *equivalent to* g_1 if there exist isometries, h_1 from g_0 to d, and h_2 from g_1 to d, such that h_1 is isotopic to h_2 relative to $\partial\mathbb{D}$. (In fact if h_t is an isotopy from h_1 to h_2 then we can pull back hyperbolic metrics g_t which represent a deformation of g_0 into g_1.) We refer to an equivalence class of relative hyperbolic metrics on $(\mathbb{D}, d)$ as a *relative hyperbolic structure on* $(\mathbb{D}, d)$.

Often we wish to consider a more restricted set of relative hyperbolic metrics on $\mathbb{D}$; metrics, in a sense, closer to the reference metric d. Let $\mathcal{M}$ denote the group of homeomorphisms of $\partial\mathbb{D}$ which arise as restrictions of the directly conformal Möbius transformations preserving $\mathbb{D}$. Let $\mathcal{F}$ be any group of orientation preserving homeomorphisms of $\partial\mathbb{D}$ which contains $\mathcal{M}$ as a subgroup. We call g an *$\mathcal{F}$-relative hyperbolic metric on* $\mathbb{D}$ if there exists a homeomorphism h of $\bar{\mathbb{D}}$ such that $h|_{\mathbb{D}}$ is an isometry from g to d, and $h|_{\partial\mathbb{D}} \in \mathcal{F}$. We define an *$\mathcal{F}$-relative hyperbolic structure on* $\mathbb{D}$ to be an equivalence class of $\mathcal{F}$-relative hyperbolic metrics on $\mathbb{D}$.

We show how to obtain a characterisation of the set of $\mathcal{F}$-relative hyperbolic structures on $(\mathbb{D}, d)$. There is a well defined map from the set of $\mathcal{F}$-relative hyperbolic structures on $\mathbb{D}$ to the set of right cosets of $\mathcal{M}$ in $\mathcal{F}$ given as follows: map the class $[g]$ to the coset $\mathcal{M} \circ h|_{\partial\mathbb{D}}$, where h is any isometry from g to d.

Theorem 1.1 *The map defined above is a bijection.*

The proof is straightforward in view of the fact that any homeomorphism of $\partial\mathbb{D}$ can be extended to a homeomorphism of $\mathbb{D}$, and this extension is unique up to isotopy relative to infinity.

Let us now extend the definitions we have made so far, to an arbitrary complete hyperbolic surface (F, ρ). We can assume that $(\mathbb{D}, d)$ is the universal cover of (F, ρ). An *$\mathcal{F}$-relative hyperbolic metric on* F is a metric on F which lifts to an $\mathcal{F}$-relative hyperbolic metric on $\mathbb{D}$.

Let h_0 and h_1 be isotopic homeomorphisms of F. We call h_t an *isotopy relative to infinity* if it lifts to isotopy of $\mathbb{D}$ relative to $\partial\mathbb{D}$. Metrics, g_0 and g_1 on F, are *equivalent* if there is an isometry h from g_0 to g_1 which is isotopic to the identity relative to infinity. If $(F, \rho) = (\mathbb{D}, d)$, this agrees with the earlier definition. An *$\mathcal{F}$-relative hyperbolic structure* on F is an equivalence class of $\mathcal{F}$-relative hyperbolic metrics on F.

The following definition is due to Ahlfors (see [1]). A map $f : \mathbb{R} \to \mathbb{R}$ is *K-quasisymmetric* ($K \geq 1$) if it is an order preserving homeomorphism of $\mathbb{R}$, and satisfies

$$K^{-1} \leq \frac{f(x+t) - f(x)}{f(x) - f(x-t)} \leq K,$$

for all $x \in \mathbb{R}$ and $t > 0$. If f is K-quasisymmetric and f_1 and f_2 are Möbius transformations of $\mathbb{R}$ which preserve order and fix ∞ then $f_1 \circ f \circ f_2$ is clearly also K-quasisymmetric.

A map of the circle $\partial\mathbb{D}$ is *K-quasisymmetric* if we can transform it into a K-quasisymmetric map of $\mathbb{R}$ by composing it with appropriate Möbius transformations of $\mathbb{C}$. A map is *quasisymmetric* if it K-quasisymmetric for some $K \geq 1$.

Later we will use an equivalent, more natural, definition of quasisymmetry given in terms of cross ratios. Thurston's definition in [6], although equivalent to Ahlfors', is not used here. Quasisymmetric relative hyperbolic structures will be of particular interest.

We now proceed toward the definition of a left earthquake map. A *lamination* λ on (F,g) is a set of disjoint simple geodesics (ie. without transverse self intersections) such that $\bigcup\lambda$ is a closed subset of F. An element of λ is called a *leaf*. A component of $F - \bigcup\lambda$ is called a *gap*. A *stratum* of the lamination is either a leaf or a gap. For each point $x \in F$ we denote by $\hat{x}$ the stratum containing x.

Let $(\mathbb{D}_1, d), (\mathbb{D}_2, d)$ be two oriented copies of the Poincaré disk. Let λ be a lamination on $\mathbb{D}_1$. Let $E : \mathbb{D}_1 \to \mathbb{D}_2$ be a map which restricts to an orientation preserving isometry on each stratum of λ. Let A be a stratum of λ. Denote by $(E|A) : \mathbb{D}_1 \to \mathbb{D}_2$ the unique orientation preserving isometry which agrees with E on A. Denote by $\mathrm{cmp}_E(A,B)$ the isometry $(E|A)^{-1} \circ (E|B) : \mathbb{D}_1 \to \mathbb{D}_1$, where A and B are strata of λ. Suppose that $\mathrm{cmp}_E(A,B)$ is hyperbolic with axis weakly separating A from B. We say that $\mathrm{cmp}_E(A,B)$ *maps to the left* if, viewed from the repelling fixed point of $\mathrm{cmp}_E(A,B)$, A lies to the left of B.

A map $E : \mathbb{D}_1 \to \mathbb{D}_2$ is a *λ-left earthquake map* if

1. The restriction of E to any stratum of λ is the restriction of an orientation preserving isometry from $\mathbb{D}_1$ to $\mathbb{D}_2$.

2. Whenever $\mathrm{cmp}_E(A,B)$ is non-trivial, it is hyperbolic, its axis weakly separates A from B, and it maps to the left.

Condition 2 implies that E is injective. We differ from Thurston here in that we do not require any of the comparison isometries of E to be non-trivial, nor do we require that E be surjective. See Figure 1.

An earthquake between hyperbolic surfaces is defined as an earthquake between universal covers which commutes with the covering transformations. An earthquake may be interpreted geometrically as cutting and gluing to get a surface of a different shape.

Let E be a λ-left earthquake map. Let l be a leaf in λ. Denote by $x \nearrow l$ and $x \searrow l$, convergent sequences of points approaching l from one side or the other. The limits $\lim_{x\nearrow l}(E|\hat{x})$ and $\lim_{x\searrow l}(E|\hat{x})$ exist (for reasons which will become apparent in Section 2) and differ, at most, by a hyperbolic isometry with axis l. We can alter E on the leaf l by pre-composing $E|l$ with a translation along l. As long as $(E|l)$ lies between the above limits, E

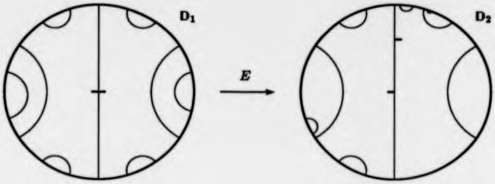


Figure 1: ***An example of a left earthquake E with finite lamination.***

is still a left earthquake. We obtain a *leaf variant* of E by altering E in this way on any or all leaves at which E is discontinuous. See Figure 2.

Thurston proves in [6] that every surjective earthquake $E : \mathbb{D} \to \mathbb{D}$ has a well defined extension to $\partial\mathbb{D}$ which is a homeomorphism. Earthquakes which differ only by leaf variance take the same boundary values. Thus we have a map from equivalence classes, under leaf variance, of surjective left earthquakes, into homeomorphisms of $\partial\mathbb{D}$. The main theorem proved by Thurston in [6] says that this correspondence is a bijection.

In view of Theorem 1.1 and the above correspondence we can regard the relative hyperbolic structures on $\mathbb{D}$ as the equivalence classes of surjective left earthquakes up to leaf variance and post-composition with isometries of $\mathbb{D}$.

We use $|T|$ to denote the distance that a hyperbolic isometry T moves points on its axis. A λ-left earthquake E is *uniformly bounded* if there exist positive constants η and M such that $d(A, B) \leq \eta \Rightarrow |\mathrm{cmp}_E(A, B)| \leq M$ for all strata A, B of λ.

In section 3 we show how to approximate a uniformly bounded earthquake by a map of the following kind. A map $\Phi : \mathbb{D} \to \mathbb{D}$ is *K-quasi-isometric* ($K \geq 1$) if it satisfies

$$K^{-1} d(x, y) \leq d(\Phi(x), \Phi(y)) \leq K d(x, y)$$

for all $x, y \in \mathbb{D}$.

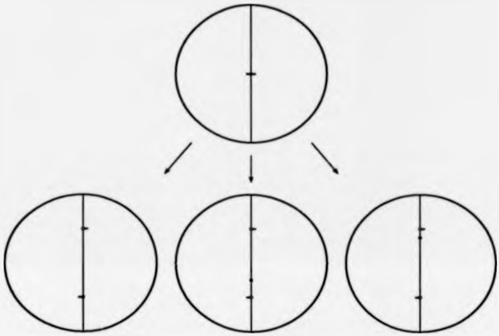


Figure 2: *A left earthquake with only one leaf in its lamination is called 'elementary'. Three leaf variants of an elementary left earthquake are shown here.*

A *transverse metric* μ on a geodesic lamination λ of H^2 is a pseudometric on the set of strata of λ with the following additional property. For all strata A, B and C of λ such that C separates A from B, $\mu(A,B) = \mu(A,C) + \mu(C,B)$. (A pseudometric is symmetric and satisfies the triangle inequality but is not necessarily non-zero on pairs of distinct elements.) Given points $x, y \in \mathrm{H}^2$ we write $\mu(x,y)$ instead of $\mu(\bar{x},\bar{y})$.

A transverse metric μ is *uniformly bounded* if there exist positive constants η and M such that $d(x,y) \leq \eta \Rightarrow \mu(x,y) \leq M$ for all $x, y \in \mathrm{H}^2$.

A *transverse measure* on a geodesic lamination λ of H^2 assigns a positive regular Borel measure m to each geodesic interval in H^2 with the following additional properties. If J is a subinterval of I then J carries the measure induced from I. If the strata containing the endpoints of I are the same as the strata containing the endpoints of J then $m(I) = m(J)$. We define uniform boundedness in exactly the same way as for a transverse metric.

A metrized/measured lamination *on a surface* is a metrized/measured lamination on the universal cover which is invariant under the group of covering transformations. We will show in Section 2.4 how a single left earthquake corresponds to a metrized lamination while an earthquake, together with all its leaf variants, corresponds to a measured lamination.

We explain now how the conformal structures on a surface are related to the quasisymmetric relative hyperbolic structures.

Let F be a Riemann surface whose universal cover is conformally equivalent to D. Since the Poincaré metric on D is invariant under Möbius transformations of D, it projects to a complete hyperbolic metric on F. Thus complete hyperbolic metrics on F and conformal structures on F are canonically interchangeable. Beurling and Ahlfors [3] have shown that any quasisymmetric map of ∂D can be extended to a quasiconformal map of D and conversely that any quasiconformal map of D has quasisymmetric boundary values. It follows from these facts that Bers' definition of the Teichmüller space of D (see [2]) is equivalent to Thurston's definition of the space of quasisymmetric relative hyperbolic structures on D. Douady and Earle [4] give a more natural way of extending a quasisymmetric map of the circle to a map of the disc. From their result it follows that the two spaces noted above coincide for any complete hyperbolic surface.

1.2 Summary of content

Thurston states in [6], and we prove here, that it is precisely the uniformly bounded earthquakes which have quasisymmetric boundary mappings. This

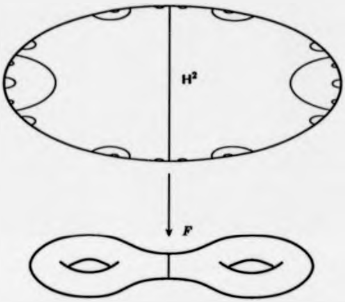


Figure 3: *A lamination on a surface F is equivalent to a collection of disjoint simple geodesics on the surface whose locus is closed. An example of this is shown here.*

gives us a bijection between equivalence classes of uniformly bounded earthquakes $\mathcal{M} \circ [E]$ on a surface F and the quasisymmetric relative hyperbolic structures on F.

A process somewhat like integration allows us to obtain a left earthquake from a metrised lamination. Infinitesimally the metric says how big the comparison isometry between nearby strata is. This gives a bijection between metrised laminations and cosets $\mathcal{M} \circ E$. A transverse measure m on a lamination gives rise to a metric μ by setting $\mu(x, y) = (m[x, y) + m(x, y])/2$. The space of measured laminations is bijective with the set of equivalence classes of earthquake maps $\mathcal{M} \circ [E]$. If we begin with a metrised lamination on a surface we obtain an earthquake between surfaces.

Uniformly bounded metrics/measures correspond to uniformly bounded earthquakes. From this we obtain a canonical bijection between the space of uniformly bounded measured laminations on a surface and the space of quasisymmetric relative hyperbolic structures on the same surface.

Douady and Earle have shown in [4] that quasisymmetric maps of the circle extend naturally to quasi-isometries of the disk. Thurston gives a parallel result, which we prove here, namely that any uniformly bounded earthquake can be approximated by a quasi-isometric diffeomorphism in a natural way. The approximation has the same map at infinity as the earthquake. By 'natural' we mean that, if we begin with a map between the universal covers of two surfaces which commutes with the covering transformations, then we obtain a map with the same property. That is, we obtain a map between the two surfaces. This gives a more direct way of seeing a uniformly bounded earthquake as a map between hyperbolic metrics.

Let p and q be quasisymmetric relative hyperbolic structures on a surface. By the characterisation of relative hyperbolic structures, given by Theorem 1.1, there is a quasisymmetric homeomorphism of the circle which relates p to q. Thurston's earthquake theorem implies that we can extend this map to a uniformly bounded left earthquake. The earthquake gives us a uniformly bounded metrised lamination. Multiplying the metric by a real parameter in the range $[0, 1]$ gives a path of metrised laminations. The corresponding path of earthquakes beginning at the identity gives a path in the space of hyperbolic structures beginning at p and ending at q. Such paths of earthquakes were used by Kerckhoff [5] in his famous paper solving the Nielsen problem.

2 Metrized laminations and earthquakes

2.1 The source lamination

Let λ be a geodesic lamination on $\mathbb{H}^2$ and let E be a λ-left earthquake. Let W be the union of all the open subsets of $\mathbb{H}^2$ on which E restricts to an isometry. Clearly points not in λ belong to W. Let γ be a leaf of λ. If γ intersects with W then γ is wholly contained in W. We deduce that the complement of W is a sublamination of λ. We call this the *source lamination* of E.

Let x and y be points in the same component of W. By compactness the interval $[x, y]$ is covered by finitely many open sets on which E restricts to an isometry. We deduce that E restricts to the same isometry in a neighbourhood of y as in a neighbourhood of x. Therefore E restricts to an isometry on each stratum of the source lamination.

Let A and B be strata in the source lamination of E. Let X and Y be strata of λ such that $X \subseteq A$ and $Y \subseteq B$. Then $\text{cmp}(A, B) = \text{cmp}(X, Y)$ so whenever this is non-trivial, its axis weakly separates X from Y. Since this holds for all such X and Y, the axis of $\text{cmp}(A, B)$ weakly separates A from B.

Let us write λ' for the source lamination of E. We have shown that E is a λ'-left earthquake. We also have the following non-triviality property, namely if A and B are separated by any leaf of λ' then $\text{cmp}(A, B)$ is not the identity.

To prove this, we suppose that $\text{cmp}(A, B)$ is trivial. Let C be any stratum which weakly separates A from B. Clearly $\text{cmp}(A, C)$ must be trivial. Therefore E restricts to an isometry on the union of all strata which weakly separate A from B. Any leaf of λ which separates A from B intersects with the interior of this union. Therefore such a leaf does not appear in λ'.

2.2 The shearing metric

An isometry of the hyperbolic plane may be classified as elliptic, parabolic or hyperbolic. To avoid confusion caused by different meanings of the word 'hyperbolic', we will always refer to an isometry which is hyperbolic as a translation. The following lemma is standard.

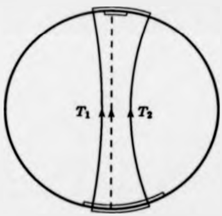
Lemma 2.1 *If T is a distance t translation along a geodesic γ, and x is any point of $\mathbb{H}^2$ then $d(x, Tx) \leq t \cosh d(x, \gamma)$.*

Lemma 2.2 *Let $\{\gamma_1, \ldots, \gamma_n\}$ be a set of disjoint geodesics in $\mathbb{H}^2$ which cross a geodesic segment α of length ϵ in order of subscript. Let T_i be a distance t_i translation with axis γ_i, and suppose that all translations are in the same direction (ie. all the attracting endpoints lie to the same side of α). Let T be the composition $T_1 \circ \ldots \circ T_n$. Then T is a translation. Let the translation distance of T be t and let the axis be γ. Then γ weakly separates γ_1 from γ_n and*

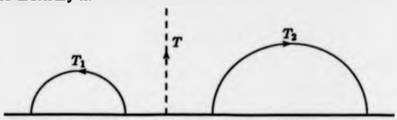
$$(t_1 + \ldots + t_n) \leq t \leq (t_1 + \ldots + t_n)\cosh \epsilon.$$

Proof: We will prove by induction that T is hyperbolic, that γ separates γ_1 from γ_n and that the left hand inequality holds. We therefore restrict our attention to the case $n = 2$.

The following argument, due to Thurston, uses the Poincare disc model for $\mathbb{H}^2$. The region between γ_1 and γ_2 intersects with S^1_∞ in two disjoint closed intervals. The interval between the attracting endpoints of γ_1 and γ_2 is mapped into itself by T. Therefore T has a fixed point in this interval. The interval between the repelling endpoints of γ_1 and γ_2 is mapped over itself by T. Therefore T has a fixed point in this interval also. Since T has two fixed points it is a translation. Since γ joins these two fixed points it separates γ_1 from γ_2.



Now choose coordinates in the upper half plane so that γ is the positive imaginary axis and T is an enlargement with factor e^t. The derivative of T_2 at 0 is at least e^{t_2}. The derivative of T_1 at $T_1^{-1}(0)$ is at least e^{t_1}. Since $T_1^{-1}(0) = T_2(0)$ we deduce that the derivative of T at 0 is at least $e^{t_1}e^{t_2}$. This implies $t \geq t_1 + t_2$. Now apply induction to generalise the results obtained so far to arbitrary n.



We now prove the second inequality. Let x be the point $\alpha \cap \gamma$. Since x lies on γ, $d(x, Tx) = t$. Since $d(x, \gamma_i) \leq \epsilon$, Lemma 2.1 implies $d(x, T_i x) \leq t_i \cosh \epsilon$. $T_1 \circ \ldots \circ T_{i-1}$ is an isometry so

$$d(T_1 \ldots T_{i-1} x, T_1 \ldots T_i x) \leq t_i \cosh \epsilon.$$

Therefore

$$t \leq (t_1 + \ldots + t_n) \cosh \epsilon. \qquad \square$$

We now fix a left earthquake, E say, on $\mathbf{H}^2$ and show that it gives rise to a well defined transverse metric on its source lamination. Given a geodesic segment I and a partition $P = \{x_0, x_1, \ldots, x_n\}$ of I, denote by $\Sigma\mathrm{SL}(I,P)$ the sum of shearing lengths,

$$\Sigma\mathrm{SL}(I,P) = \sum_{i=1}^{n} |\mathrm{cmp}(x_{i-1}, x_i)|$$

Set $T = \mathrm{cmp}(x_1, x_n)$ and $T_i = \mathrm{cmp}(x_{i-1}, x_i)$. Let $\epsilon = \mathrm{mesh}(P)$ and let P' be any refinement of P. For each interval $[x_{i-1}, x_i]$, Lemma 2.2 implies that

$$||T_i| - \Sigma\mathrm{SL}([x_{i-1}, x_i], P'|_{[x_{i-1}, x_i]})| \leq |T_i|(\cosh \epsilon - 1)$$

and hence

$$|\Sigma\mathrm{SL}(I,P) - \Sigma\mathrm{SL}(I,P')| \leq O(|T|\epsilon^2).$$

This proves that $\Sigma\mathrm{SL}(I,P)$ converges to a well defined limit as the mesh of P tends to zero. Denote this limit by $\mu(x,y)$, where x and y denote the endpoints of I. Lemma 2.2 implies that

$$\mu(x,y) \leq |T| \leq \mu(x,y) \cosh d(x,y). \qquad (1)$$

We call μ the *shearing metric* of E. It is clear from the construction of μ that it is a transverse metric on the source lamination of E.

Since μ is bounded on any bounded subset of $\mathbf{H}^2$ (see Figure 4) we deduce that the image, under E, of any bounded set is again bounded. Estimate 1 also implies that E is uniformly bounded if and only if μ is.

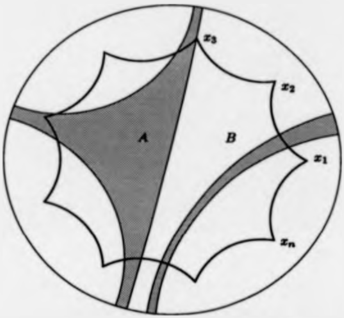


Figure 4: *We have $2\mu(A,B) \leq \mu(x_1,x_2) + \ldots + \mu(x_{n-1},x_n) + \mu(x_n,x_1)$. In general, μ is bounded by half the right-hand side on all strata which intersect with the polygon.*

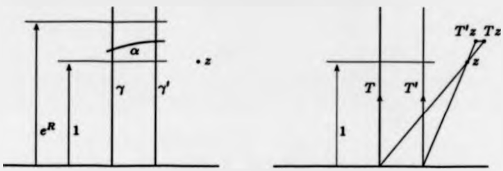
2.3 Earthquakes from metrized laminations

We prove in this section that to each metrized lamination (λ, μ) there corresponds a λ-left earthquake with shearing metric μ and that any two such earthquake maps differ only by post-composition with an isometry.

Lemma 2.3 *Let T and T' be distance t translations, in parallel directions, along disjoint or identical axes γ and γ'. Suppose that γ is joined to γ' by a geodesic segment α, of length ϵ. Let z be a point in $\mathbb{H}^2$. Suppose we have positive constants R and K such that $\alpha \subseteq D_R(z)$ and $t \leq K$. Then*

$$d(Tz, T'z) \leq \epsilon t e^{R} e^{K}.$$

Proof: We prove the lemma first in the case where γ and γ' share a common endpoint. Choose coordinates in the upper half plane so that γ and γ' are vertical and so that z has imaginary part 1.



By the hypotheses on α, the euclidean separation of γ from γ' is at most ϵe^{R}. So the Euclidean distance of Tz from $T'z$ is at most $|e^{\pm t} - 1|\epsilon e^{R}$. Thus the hyperbolic distance of Tz from $T'z$ is at most $|1 - e^{\mp t}|\epsilon e^{R}$. This implies the lemma in the special case.

Suppose γ and γ' have no common endpoint. We introduce a third geodesic sharing one endpoint with γ and the other, with γ'. This new geodesic cuts α into two subsegments whose lengths total ϵ. Applying the above result twice completes the proof of this lemma. $\square$

Let us now fix a metrized lamination (λ, μ). Let α be a closed geodesic segment in $\mathbb{H}^2$ and let $P = \{x_0, \ldots, x_n\}$ be a partition of α. Let l denote the length of α and ϵ the mesh of P.

Definition An isometry T of $\mathbb{H}^2$, is *P-compatible* if it may be expressed as a composition $T_1 \circ \ldots \circ T_n$, in which the T_i satisfy the following.

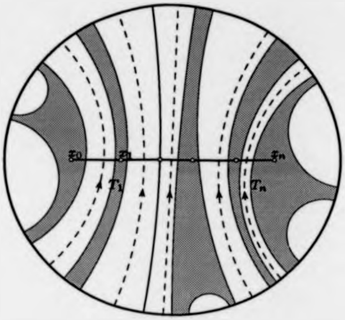


Figure 5: *For $T_{i+1} \circ \ldots \circ T_n$, as shown, to be P-compatible, it is sufficient that $|T_i|$ lies between $\mu(x_{i-1}, x_i)$ and $\mu(x_{i-1}, x_i) \cosh \epsilon$ for $i = 1, \ldots, n$.*

(i) If $\mu(x_{i-1}, x_i) = 0$ then T_i is the identity.

(ii) If $\mu(x_{i-1}, x_i) \neq 0$ then T_i translates to the left, looking from x_{i-1} to x_i, along an axis which weakly separates $\hat{x}_{i-1}$ from $\hat{x}_i$. Moreover we require that $\mu(x_{i-1}, x_i) \leq |T_i| \leq \mu(x_{i-1}, x_i) \cosh \epsilon$.

See Figure 5.

Example 1 Let T_P be the composition $T_1 \circ \ldots \circ T_n$, where the T_i are as follows.

(i) If $\mu(x_{i-1}, x_i) = 0$ then define T_i to be the identity.

(ii) If $\mu(x_{i-1}, x_i) \neq 0$ then let γ_i be the leaf in λ which crosses $[x_{i-1}, x_i]$ closest to x_{i-1}. Define T_i to be translation to the left a distance $\mu(x_{i-1}, x_i)$ along γ_i.

Example 2 Suppose that λ and μ are the source lamination and shearing metric of a left earthquake E. The comparison isometry $\text{cmp}(x_0, x_n)$

is P-compatible since it may be expressed in the form $\mathrm{cmp}(x_0, x_1) \circ \ldots \circ \mathrm{cmp}(x_{n-1}, x_n)$.

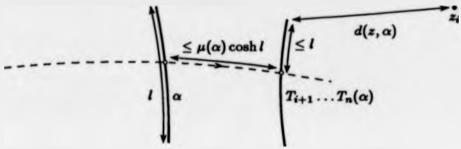
Lemma 2.4 *Let T and T' be P-compatible isometries and let x be a point of $\mathbb{H}^2$. Then we have*

$$d(Tx, T'x) \leq M\,\mathrm{mesh}(P),$$

where M is a constant depending only on $\mu(\alpha)$, l and $d(x, \alpha)$.

Proof: It is sufficient to prove this result in the special case $T = T_P$ defined in Example 1. The lemma then follows by comparing two arbitrary P-compatible isometries with T_P.

For $i = 1, \ldots, n$ let T_i be the isometries defined in Example 1. Let x_i be the point $T_{i+1} \ldots T_n x$, for $i = 0, \ldots, n-1$, and let $x_n = x$. Let $R = 2l + d(\alpha, x) + \mu(\alpha)\cosh l$. The axis of $T_{i+1} \circ \ldots \circ T_n$ crosses α and its translation length is at most $\mu(\alpha)\cosh l$.



Thus for each x_i we have

$$\alpha \subseteq D_R(x_i).$$

We proceed by finding an upper bound on $d(T_i x_i, T_i' x_i)$. Let $\epsilon = \mathrm{mesh}(P)$.

If $\mu(x_{i-1}, x_i) = 0$ then both T_i and T_i' are equal to the identity. Therefore in this case $d(T_i x_i, T_i' x_i) = 0$.

Suppose $\mu(x_{i-1}, x_i) \neq 0$. By using the inequality $\cosh\epsilon - 1 \leq \epsilon \sinh \epsilon$ and the fact that $\epsilon \leq l$, we have

$$0 \leq |T_i'| - \mu(x_{i-1}, x_i) \leq \epsilon\mu(x_{i-1}, x_i)\sinh l.$$

Let T_i'' be a distance $\mu(x_{i-1}, x_i)$ translation along the axis of T_i'. From Lemma 2.1 we deduce

$$d(T_i'' x_i, T_i' x_i) \leq \epsilon\mu(x_{i-1}, x_i)\sinh l \cosh R.$$

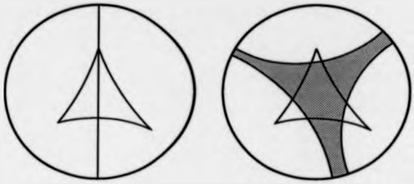


Figure 6: ***Two ways in which a stratum can intersect with all three edges of a triangle are shown here.***

Next we apply Lemma 2.3 to T_i and T_i''. Let w be the point at which the axis of T_i crosses α. Let $K = \mu(\alpha)$ and let R be as above. By definition of T_P and the definition of 'P-compatible', the axis of T_i is either disjoint from or identical to the axis of T_i''. Lemma 2.3 implies

$$d(T_i x_i, T_i'' x_i) \leq \epsilon\mu(x_{i-1}, x_i) e^R e^K .$$

Putting these together we obtain

$$d(T_i x_i, T_i' x_i) \leq \epsilon\mu(x_{i-1}, x_i)(e^R e^K + \sinh l \cosh R).$$

Applying $T_1' \circ \ldots \circ T_{i-1}'$ to both points and writing out x_i fully, this becomes

$$\begin{aligned} d(T_1' \ldots T_{i-1}' T_i T_{i+1} \ldots T_n x, T_1' \ldots T_{i-1}' T_i' T_{i+1} \ldots T_n x) \\ \leq \epsilon\mu(x_{i-1}, x_i)(e^R e^K + \sinh l \cosh R). \end{aligned}$$

Hence

$$d(Tx, T'x) \leq \epsilon\mu(\alpha)(e^R e^K + \sinh l \cosh R)$$

from which we deduce the lemma. □

Lemma 2.5 *Let λ be a geodesic lamination on $\mathbb{H}^2$. Let a, b and c be points in $\mathbb{H}^2$. We show that there is a stratum X of λ, intersecting with all three edges of the triangle a, b, c. (See Figure 6.)*

Proof: Let x be the point on $[a, b]$ closest to a such that every leaf crossing both $[a, b]$ and $[a, c]$ meets $[a, x]$. Let y be the point on $[a, b]$ closest to b such that every leaf crossing both $[a,b]$ and $[b,c]$ meets $[y,b]$. If $[a,x] \cap [y,b] \neq \emptyset$ then x intersects with all three edges and we set $X = \hat{x}$. If $[a,x] \cap [y,b] = \emptyset$ then let X be the gap containing (x,y). $\square$

Theorem 2.6 *Let (λ,μ) be a metrized lamination. There exists an earthquake with source lamination λ and shearing metric μ, and furthermore, any two such earthquakes differ only by post composition with an isometry of* $\mathbb{H}^2$.

Proof: Let A and B be strata of λ. Let α be a geodesic segment joining a point of A with a point of B. Let $P = \{x_0, \ldots, x_n\}$ be a partition of α running from A to B. Let P' be any refinement of P. Let $T_{P'}$ be defined as in Example 1. We show that $T_{P'}$ is P-compatible.

Let T_i' be the composition of the isometries making up $T_{P'}$, which take their translation distances from subintervals of $[x_{i-1}, x_i]$. If $\mu(x_{i-1}, x_i) \neq 0$ then Lemma 2.2 implies that the axis of T_i' weakly separates x_{i-1} from $\hat{x}_i$ and that

$$\mu(x_{i-1}, x_i) \leq |T_i'| \leq \mu(x_{i-1}, x_i) \cosh d(x_{i-1}, x_i).$$

This is sufficient to ensure that $T_{P'}$ is P-compatible.

Lemma 2.4 now implies that, as P is refined, T_P converges, uniformly on compact subsets of $\mathbb{H}^2$, to an isometry of $\mathbb{H}^2$. Since the limit depends only on which strata contain the endpoints of α, we will denote it by $T_{A,B}$.

Now let $T_P^{\prime} = T_1' \circ \ldots \circ T_n'$ where T_i' is translation a distance $\mu(x_{i-1}, x_i)$ along the leaf crossing $[x_{i-1}, x_i]$ closest to x_i. Lemma 2.4 implies that $T_P^{\prime}$ converges to the same limit as T_P. Let Q be P with the numbering reversed. Then $T_P^{\prime} = T_Q^{-1}$. By refining P and Q we find that $T_{A,B} = T_{B,A}^{-1}$.

If C is a stratum which weakly separates A from B then, by further subdividing each partition of α, we deduce that $T_{A,B} = T_{A,C} \circ T_{C,B}$. We wish to generalise this to the case where C does not separate A from B. By Lemma 2.5 there exists a stratum X which intersects with all three edges of the triangle a, b, c. Now $T_{A,B} = T_{A,X} \circ T_{X,B}$, $T_{B,C} = T_{B,X} \circ T_{X,C}$ and $T_{A,C} = T_{A,X} \circ T_{X,C}$. This implies that

$$T_{A,C} = T_{A,B} \circ T_{B,C}$$

for all strata A, B and C of λ. This is called the *cocycle* condition.

Let O be any stratum of λ. Let E be the map defined by $E(x) = T_{O,\hat{x}}(x)$. Clearly E restricts to an isometry on each stratum of λ. The cocycle condition implies that the comparison isometries of E are the isometries $T_{A,B}$. Since the axis of $T_{A,B}$ weakly separates A from B, and $T_{A,B}$ translates left, looking from A to B, we deduce that E is a left earthquake.

On an interval I with an ϵ-partition P we have

$$|\Sigma \mathrm{SL}(I,P) - \mu(I)| \leq \mu(I)(\cosh \epsilon - 1),$$

so E has shearing metric μ.

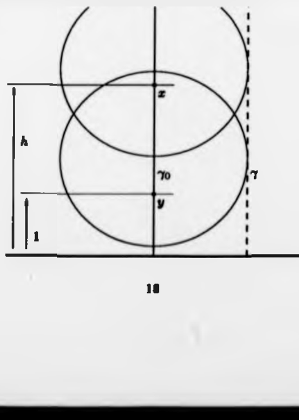
It remains to show that if E' is another earthquake with source lamination λ and shearing metric μ then E' differs from E only by post-composition with an isometry of H^2. It will be sufficient to show that the comparison isometries of E' are the same as those of E. As before let P be a partition of some geodesic segment which joins strata A and B of λ. The comparison isometries $\mathrm{cmp}_E(A,B)$ and $\mathrm{cmp}_{E'}(A,B)$ are both P-compatible, regardless of P. Therefore, by Lemma 2.4, both are equal to the limit, on refining P, of T_P. □

We now prove a few estimates in preparation for Lemma 2.8. Lemma 2.8 is a kind of continuity property for left earthquakes. It provides the basis for Theorem 2.10 in which we show that, for laminations with uniformly bounded shearing metrics, the corresponding left earthquake map is surjective.

Lemma 2.7 *Let x and y be points on a geodesic γ_0. Let γ be any geodesic disjoint from γ_0. Then*

$$\frac{d(y,\gamma)}{d(x,\gamma)} \leq e^{d(x,y)}.$$

Proof: By an appropriate choice of coordinates in the upper half plane we can assume that γ_0 occupies the position shown.



Keeping $d(x, \gamma)$ fixed we can vary γ so as to maximise $d(y, \gamma)$. This happens with γ positioned as above. The Euclidean radius of the circle with (hyperbolic) centre at x and hyperbolic radius $d(x, \gamma)$ is $h \sinh d(x, \gamma)$. Writing down the corresponding formula for the circle centered at y we obtain

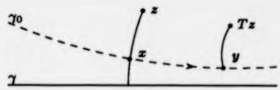
$$\sinh d(y, \gamma) = h \sinh d(x, \gamma).$$

Observe that $\sinh(t)/t$ is a monotone increasing function for $t > 0$. Therefore since $d(x, \gamma) \leq d(y, \gamma)$ we deduce that

$$\frac{d(x, \gamma)}{d(y, \gamma)} \leq \frac{\sinh d(y, \gamma)}{\sinh d(x, \gamma)} = h.$$

Since $h = e^{d(x,y)}$ the required inequality follows. $\square$

Suppose now that γ_0 is the axis of a hyperbolic isometry T. Let z be a point weakly separated from γ by γ_0. Redefine x and y as follows. Let x be the point where the perpendicular from z to γ crosses γ_0. Let $y = T(x)$.



We have

$$\frac{d(Tz, \gamma)}{d(z, \gamma)} \leq \frac{d(Tz, Tx) + d(Tx, \gamma)}{d(z, x) + d(x, \gamma)} = \frac{d(z, x) + d(y, \gamma)}{d(z, x) + d(x, \gamma)} \leq e^{|T|}$$

where the right hand inequality comes from Lemma 2.7 above. The same result for T^{-1} applied to Tz gives

$$e^{-|T|} \leq \frac{d(Tz, \gamma)}{d(z, \gamma)} \leq e^{|T|}. \tag{2}$$

Lemma 2.8 *Let E be a left earthquake with source lamination λ and shearing metric μ. Let γ be a leaf of λ, and let z be point of $\mathbb{H}^2$. Then we have*

$$e^{-\mu(z,\gamma)} \leq \frac{d(Ez, E\gamma)}{d(z, \gamma)} \leq e^{\mu(z,\gamma)}.$$

Proof: Let P be a partition of the perpendicular from x to γ. Let $T_1, \ldots, T_n$ be the comparison isometries between sucessive points in P. Repeated applications of Estimate 2 imply

$$e^{-\sum|T_i|} \leq \frac{d(Ex, E\gamma)}{d(x,\gamma)} \leq e^{\sum|T_i|}.$$

By refining P we obtain the required estimate. □

For the purposes of Lemma 2.9 and what follows we call a set of strata *bounded* if there is a point in the plane whose distance from any stratum in the set is bounded.

Lemma 2.9 *Let x, y be points in the target plane of an earthquake E. Let L be the set of leaves in γ, whose images under E intersect with $[x,y]$. If $x \in \mathrm{Im}(E)$ and L is bounded then $y \in \mathrm{Im}(E)$ also.*

Proof: Let G be a gap in the source lamination, λ say, of E. It is straightforward to check that $\partial EG \subseteq E\lambda$.

Suppose that L is empty. Let G be the stratum containing $E^{-1}x$. Since L is empty, G is a gap and ∂EG does not intersect with $[x,y]$. Therefore $y \in EG$.

Now suppose that L is non-empty. We show first that L is closed. Let l be a limit leaf of L.

By Lemma 2.8 its image El is a limit leaf of EL. So, by compactness of $[x,y]$, $El \cap [x,y]$ is non-empty. This implies $l \in L$ so L is closed.

Let γ be the leaf in L whose distance from $E^{-1}x$ is maximal. Let $s = E\gamma \cap [x,y]$. If $s = y$ then we are done. For the remainder of the proof we assume that $s \neq y$. Notice that, by our choice of γ, $(s,y] \cap E\lambda$ is empty. Let S be the set of strata of λ, separated from $E^{-1}x$ by γ. The leaf γ lies in the closure of S, either as a boundary leaf of some gap, or as a limit of leaves in

S. If γ were a limit leaf of leaves in S then, by Lemma 2.8, the image of a leaf sufficiently close to γ would cross $(x, y]$. This is impossible, so γ bounds a gap, G say, in S. Since $(x, y]$ intersects with EG but not with ∂EG, we have $y \in EG$. □

Theorem 2.10 *A left earthquake whose shearing metric is uniformly bounded is surjective.*

Proof: Let E be a left earthquake with source lamination λ and uniformly bounded shearing metric μ. Let C and η be constants of uniform boundedness on μ. Let x be a point in $\mathrm{Im}(E)$ and let y be an arbitrary point in the target plane. We show that $y \in \mathrm{Im} E$.

Let S be the set of strata of λ whose images under E intersect with $[x, y]$. Since E preserves separation, S consists of parallel strata.

Suppose μ is unbounded on S. Since μ has no discontinuity of size greater than C, we can find a sequence $\{X_n\}$ of strata in S, which satisfiy the following.

1. X_i separates X_{i-1} from X_{i+1}, and
2. $C < \mu(X_{i-1}, X_i) \leq 2C$.

Uniform boundedness implies that $d(X_{i-1}, X_i) \geq \eta$. It follows from Lemma 2.8 that $d(EX_{i-1}, EX_i) \geq \eta e^{-2C}$. Therefore $d(EX_0, EX_n) \geq n\eta e^{-2C}$. This is impossible since every stratum in ES intersects with $[x, y]$. Therefore the assumption that μ is unbounded on S is false.

Let K be an upper bound for the μ-distance of any two strata in S. Let γ be any leaf in S. Lemma 2.8 implies that

$$\begin{aligned} d(E^{-1}x, \gamma) &\leq e^K d(x, E\gamma) \\ &\leq e^K d(x, y). \end{aligned}$$

Thus the set of leaves in S is bounded and Lemma 2.9 implies $y \in \mathrm{Im} E$. □

Finally we prove a lemma concerning surjective left earthquakes.

Lemma 2.11 *Let λ be a geodesic lamination and let E be a λ-left earthquake. Suppose that E is surjective. Then $E(\lambda)$ is a lamination on the range of E.*

Proof: Since we know that $E(\lambda)$ is a set of disjoint geodesics in the range of E, it remains only to show that the locus of $E(\lambda)$ is closed. Let x' be a point in the closure of $E(\lambda)$. Let γ'_n be a sequence of geodesics in $E(\lambda)$ such that $d(\gamma'_n, x') \to 0$. Assume, without loss of generality, that γ'_n separates γ'_1 from x' for $n \geq 2$. Let γ_n and x be $E^{-1}(\gamma'_n)$ and $E^{-1}(x')$ respectively. Since E preserves separation properties, we have that γ_n separates γ_1 from x for all $n \geq 2$. It follows that $\mu(\gamma_n, x)$ is bounded by $\mu(\gamma_1, x)$. By Lemma 2.8 we see that $d(\gamma_n, x) \to 0$. Therefore $x \in \lambda$ which implies $x' \in E(\lambda)$. □

As a consequence of this lemma we have the following. If E is a surjective λ-left earthquake then E^{-1} is an $E(\lambda)$-*right* earthquake.

2.4 Measured laminations and leaf variance

In this section we show that, just as there is a correspondence between metrized laminations and earthquakes, there is a similar correspondence between measured laminations and equivalence classes of earthquakes under leaf variance. (Lemma 2.12 will imply that leaf variance gives an equivalence relation.)

We begin by showing how to obtain a transverse measure from a transverse metric.

Let μ be a transverse metric on a lamination. Define the measure m of a half-open geodesic interval $[x, y)$ to be

$$m[x, y) = \lim_{\delta \searrow 0} \mu(x - \delta, y - \delta)$$

where $x - \delta$ denotes a point at distance δ from x on the geodesic through x and y etc. Since $\mu(x - \delta, x)$ and $\mu(x, y - \delta)$ are monotone functions of δ the above limit certainly exists.

The fact that $m[x, y)$ depends only on which strata contain x and y is clear. We show next that m is additive on any partition of $[x, y)$ into countably many half-open subintervals. It will then follow from standard measure theory that m defines a regular positive Borel measure on each geodesic in $\mathbf{H}^2$ and is thus a transverse measure.

Let $[x_n, y_n)$ be any sequence of half open intervals which partitions $[x, y)$. It is easy to show that m is additive on finite partitions of $[x, y)$. It follows that

$$\sum_{n=1}^{\infty} m[x_n, y_n) \leq m[x, y).$$

We obtain the reverse inequality to show that equality holds.

Fix any $\epsilon > 0$. For each n choose $\delta_n > 0$ such that

$$|\mu(x_n - \delta, y_n - \delta') - m[x_n, y_n)| < \epsilon 2^{-n}$$

for all $\delta, \delta' \leq \delta_n$. The intervals $(x_n - \delta_n, y_n)$ form an open cover of $[x-\nu, y-\nu]$ for ν sufficiently small. In particular we may choose ν such that

$$|\mu(x - \nu, y - \nu) - m[x, y)| < \epsilon.$$

By compactness there exists N such that $(x_1 - \delta_1, y_1), \ldots, (x_N - \delta_N, y_N)$ cover $[x - \nu, y - \nu]$. Now choose $\delta > 0$ sufficiently small that the intervals $(x_1 - \delta_1, y_1 - \delta), \ldots, (x_N - \delta_N, y_N - \delta)$ still cover $[x - \nu, y - \nu]$ and $\delta \leq \delta_1, \ldots, \delta_N$. We have

$$\begin{aligned}
\sum_{n=1}^{\infty} m[x_n, y_n) &\geq \sum_{n=1}^{N} m[x_n, y_n) \\
&\geq \sum_{n=1}^{N} \mu(x_n - \delta_n, y_n - \delta) - \epsilon \\
&\geq \mu(x - \nu, y - \nu) - \epsilon \\
&\geq m[x, y) - 2\epsilon.
\end{aligned}$$

Since ϵ was arbitrary we have shown that

$$\sum_{n=1}^{\infty} m[x_n, y_n) \geq m[x, y).$$

It follows that m is countably additive and therefore defines a measure.

Let E be a left earthquake with source lamination λ. Let μ the shearing metric of E and m the corresponding transverse measure. Define λ_d to be the set of leaves $l \in \lambda$ where

$$\lim_{x \nearrow l}(E|x) \neq \lim_{x \searrow l}(E|x).$$

Lemma 2.12 *The set λ_d is countable and equal to the set of leaves where m has an atom. Let E' be a left earthquake which agrees with E off λ_d. Then E' is a leaf variant of E and has shearing measure m.*

Proof: Let l be a leaf in λ and let I be an interval crossing l. Let x be the point $l \cap I$. From the definition of m we see that

$$m(x) = \lim_{\delta \searrow 0} \mu(x - \delta, x + \delta).$$

It follows that

$$m(x) = |\lim_{x\nearrow l}(E|x)^{-1} \circ \lim_{x\searrow l}(E|x)|.$$

Now let $\{I_n\}$ be a sequence of intervals in $\mathbb{H}^2$, each having finite length, such that every geodesic in $\mathbb{H}^2$ crosses at least one interval in the sequence. Since $m(I_n)$ is finite for each n, m may have at most countably many atoms on each interval. This proves the first part of the lemma.

Let l be a leaf in λ. Since E and E' are equal on $\mathbb{H}^2 - \bigcup \lambda_d$ and the latter is dense in $\mathbb{H}^2$,

$$\lim_{x\nearrow l}(E|x) = \lim_{x\nearrow l}(E'|x).$$

Isometries $(E|l)$ and $(E'|l)$ differ from the above limit at most by a hyperbolic isometry with axis l. Since E and E' are left earthquakes, $(E|l)$ and $(E'|l)$ both lie between

$$\lim_{x\nearrow l}(E|x) \text{ and } \lim_{x\searrow l}(E|x).$$

Hence E and E' are leaf variants.

Let μ' denote the shearing metric of E' and m' the corresponding measure. Let I be a geodesic interval whose endpoints miss λ_d. By taking partitions of I which miss λ_d we deduce that $\mu(I) = \mu'(I)$. Finally, let $[x, y)$ be any half-open geodesic interval. By choosing $\delta \searrow 0$ such that the endpoints of $[x - \delta, y - \delta]$ miss λ_d we find, as required, that $m[x, y) = m'[x, y)$. □

Since by definition leaf variants of E agree off λ_d we can deduce immediately that leaf variance gives an equivalence relation. Moreover we have shown that equivalent earthquakes have the same shearing measure.

For the converse let E and E' be earthquakes whose shearing metrics μ and μ' give rise to the same transverse measure m. Let λ_d be defined as before. Let $[x, y]$ be an interval whose endpoints miss λ_d. For such an interval $\mu(x, y) = m[x, y] = \mu'(x, y)$. Let P be a partition of $[x, y]$ which misses λ_d. Applying Lemma 2.4 to $\mathrm{cmp}_E(x, y)$ and $\mathrm{cmp}_{E'}(x, y)$ we deduce that, off λ_d, E and E' have the same comparison isometries. This implies that E and E' differ at most by leaf variance and post-composition with an isometry.

2.5 Earthquakes on surfaces

Let E be a *surjective* left earthquake on $\mathbb{H}^2$ with source lamination λ and shearing metric μ. Let Γ be a Fuchsian group acting on the domain of E. Suppose that (λ, μ) is invariant under Γ.

Lemma 2.13 *The maps $E \circ \Gamma \circ E^{-1}$ form a Fuchsian group acting on the range of E.*

Proof: We begin by showing that for each $g \in \Gamma$ the map $E \circ g \circ E^{-1}$ is an isometry. Let A and B be strata of λ, α a geodesic segment linking them and P a partition of α. Define T_P as in Example 1 preceding Lemma 2.4. An element $g \in \Gamma$ gives us strata gA and gB and an interval $g\alpha$ with partition gP. By the invariance of (λ, μ) we have $T_{gP} = g \circ T_P \circ g^{-1}$. On refining P we obtain $\mathrm{cmp}(gA, gB) = g \circ \mathrm{cmp}(A, B) \circ g^{-1}$. Therefore

$$(E|gA)^{-1} \circ (E|gB) = g \circ (E|A)^{-1} \circ (E|B) \circ g^{-1}$$

which implies that

$$(E|gB) \circ g \circ (E|B)^{-1} = (E|gA) \circ g \circ (E|A)^{-1}.$$

Thus $E \circ g \circ E^{-1}$ gives us the same isometry regardless of which stratum of $E(\lambda)$ we consider.

Let us call the resulting group of isometries Γ'. It remains to prove that Γ' is Fuchsian.

Let x be a point of H^2. Let $\{g_n\}$ be a sequence of distinct elements of Γ. The sequence $\{g_n \circ E^{-1}x\}$ is unbounded. Since E^{-1} is a right earthquake it cannot map a bounded set to an unbounded one. It follows that $\{E \circ g_n \circ E^{-1}x\}$ has no finite limit point. □

Let F be a hyperbolic surface and (λ, μ) a uniformly bounded metrised lamination on F. Lemma 2.13 implies that there is a second hyperbolic surface F' and an earthquake $E : F \to F'$ such that E has shearing metric μ. If F is compact then every metrised lamination on F is uniformly bounded. If F has finite area then (λ, μ) will be uniformly bounded precisely when λ does not enter any ideal puncture of F.

2.6 Density of simple left earthquakes

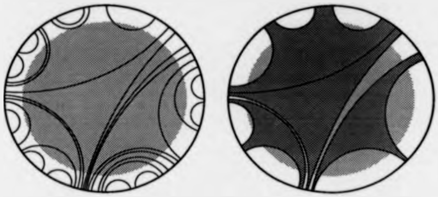
A left earthquake is called *simple* if its source lamination is finite. Here we give two slightly different ways of approximating a left earthquake by a simple left earthquake.

We prove a number of preparatory lemmas first. Let λ be a lamination and K a compact convex subset of H^2. Write $S(\lambda, K)$ for the set of strata of λ which intersect with K. Let

$$\mathcal{T}_0 = \{X \in S(\lambda, K) | K - X \text{ has three or more components}\}.$$

Suppose that $\mathcal{T}_0$ contains at least n strata. One can deduce the existence of at least $n+2$ disjoint half-planes which intersect with K. Therefore $\mathcal{T}_0$ has to be finite.

Here is a diagram which illustrates the above definition.



On the left is shown a lamination λ and a compact convex set K. On the right is shown the resulting set of strata $\mathcal{T}_0$.

Lemma 2.14 *Let $\mathcal{T}$ be any finite subset of $\mathcal{S}(\lambda, K)$ containing $\mathcal{T}_0$. Each component of $\mathrm{H}^2 - \mathcal{T}$ is bounded by at most two strata in $\mathcal{T}$.*

Proof: Let W be a component of $\mathrm{H}^2 - \mathcal{T}$. Suppose three strata bound W. Choose a point in each and join them up to form a triangle. By Lemma 2.5 there is a stratum A of λ which intersects with all three edges of this triangle. It is clear that A is a member of $\mathcal{T}_0$ and yet A is contained in W. Since this is impossible at most two strata in $\mathcal{T}$ bound W. □

Theorem 2.15 *Let E be a λ-left earthquake. Let $\mathcal{T}$ be a finite set of strata of λ. There exists a simple left earthquake F which agrees with E on strata in $\mathcal{T}$.*

Proof: Let K be any compact convex subset of H^2 which intersects with all the strata in $\mathcal{T}$. Enlarge $\mathcal{T}$ if necessary so that it contains $\mathcal{T}_0$ (defined above). Define a set of geodesics Λ as follows. Let Λ contain the axis of $\mathrm{cmp}(A, B)$ whenever the latter is non-trivial and $A, B \in \mathcal{T}$ are not separated by any stratum in $\mathcal{T}$.

Let γ be a geodesic in Λ coming from A, B as above. If γ intersects with A then A is a leaf and $\gamma = A$. Suppose γ does not intersect with either A or

B. Then γ intersects with a component W of $\mathbf{H}^2 - \mathcal{T}$ bounded by A and B. Since, by the above lemma, A and B are the only strata of $\mathcal{T}$ which bound W, γ is contained in W.

We have shown that each geodesic in Λ is either a leaf in $\mathcal{T}$ or is contained in a component of $\mathbf{H}^2 - \mathcal{T}$. Moreover it is clear that at most one such geodesic can lie in any given component of $\mathbf{H}^2 - \mathcal{T}$. We deduce from this that the geodesics in Λ are disjoint and so form a finite lamination.

Now we define a transverse metric ν on Λ. We show how ν is defined for a single leaf γ of Λ. From this we are able to work out the values of Λ on the whole plane. Let $A, B \in \mathcal{T}$ be strata which give rise to γ as above.

If neither A nor B is equal to γ, define the ν-distance from γ to each side of γ to be $\|\mathrm{cmp}(A,B)\|/2$.

If, without loss of generality, $A = \gamma$, define the ν-distance from γ to the side of γ containing B to be $\|\mathrm{cmp}(A,B)\|$. If any other pair of strata gives rise to γ, it will be of the form A, B' with $A = \gamma$, and B' lying on the other side of γ from B. We then define the ν-distance from γ to the side of γ containing B' to be $\|\mathrm{cmp}(A,B')\|$. If no other such pair of strata exists, define this ν-distance to be 0.

Choose a stratum Z in $\mathcal{T}$ and define F to be the Λ-left earthquake with shearing metric ν which agrees with E on Z. Let X, Y be strata of $\mathcal{T}$. Let A be the stratum of Λ containing X and B the stratum containing Y. It is not hard to check that $\mathrm{cmp}_E(X,Y) = \mathrm{cmp}_F(A,B)$. This completes the proof of our Theorem. $\square$

Next we prove a second, more detailed theorem.

Theorem 2.16 *Let E be a left earthquake. On each compact subset K of $\mathbf{H}^2$ we can approximate E uniformly by a simple left earthquake. Moreover if the source lamination of E is uniformly bounded with constants M and η then we can approximate E by an earthquake whose source lamination is uniformly bounded with constants $M + 2\epsilon$ and η, where ϵ may be arbitrarily small.*

Define

$$\Lambda_0 = \{\gamma \in \lambda | \gamma \cap K \neq 0 \text{ and } \gamma \subseteq \partial X \text{ for some } X \in \mathcal{T}_0\}.$$

Notice that it is quite possible for Λ_0 to be empty.

Given any sublamination Λ of λ and stratum $A \in S(\Lambda, K)$ define

$$A^* = \{X \in S(\lambda, K) | X \subseteq A\}.$$

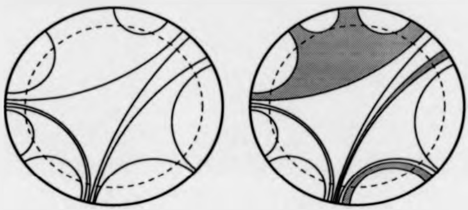
Let Λ be any finite sublamination of λ which contains Λ_0 and contains only leaves of λ which intersect with K.

Lemma 2.17 *Let $A \in S(\Lambda, K)$ and $X \in A^*$. Then*

1. *X separates components of $K - A$ pairwise and*
2. *if $K - A$ has three or more components then $A^* = \{X\}$.*

Proof: Draw pictures of all the possibilities which the lemma excludes. It should become apparent why they cannot occur. A formally correct proof of this lemma, although not hard to construct would not be very illuminating. □

Here is a diagram illustrating the two cases which appear in the above lemma.



On the left is the lamination Λ_0 arising from the lamination and convex set shown in the previous diagram. On the right, three strata of λ are shown superimposed.

Lemma 2.18 *Let $A \in S(\Lambda, K)$ and let $X, Y \in A^*$. If $X \neq Y$ then either*

1. *∂A contains only one leaf and either X separates Y from ∂A or Y separates X from ∂A or*
2. *∂A contains two leaves, one of which is separated from X by Y.*

Proof: As for the previous lemma. □

Suppose now that λ is metrised with transverse metric μ. Given strata $X, Y \in S(\lambda, K)$ define

$$d_K(X,Y) = \inf\{d(x,y) | x \in X \cap K, y \in Y \cap K\}.$$

Given a leaf $\gamma \in \lambda$ and any stratum X of λ define

$$\overline{\mu}(X,\gamma) = \sup\{\mu(X,Y) | Y = X \text{ or } Y \text{ separates } X \text{ from } \gamma\}.$$

Fix $\epsilon > 0$. Suppose that γ intersects with K. Define

$$N_\epsilon(\gamma) = \bigcup\{X \in S(\lambda, K) | d_K(X,\gamma) \leq \epsilon \text{ and } \overline{\mu}(X,\gamma) \leq \epsilon\}.$$

Since $\overline{\mu}(X,\gamma) \to 0$ as $d_K(X,\gamma) \to 0$, each point of $\gamma \cap K$ is an interior point of $N_\epsilon(\gamma) \cap K$ with respect to the subspace topology on K. Define $U_\epsilon(\gamma)$ to be the interior of $N_\epsilon(\gamma)$ in the subspace topology on K.

We summarise the properties of $N_\epsilon(\gamma)$ and $U_\epsilon(\gamma)$ which we will use in the proof of the following Lemma. Let $X \in S(\lambda, K)$ be a gap and $\gamma \in S(\lambda, K)$ be a leaf. If $X \subseteq N_\epsilon(\gamma)$ and $Y \in S(\lambda, K)$ separates X from γ then $Y \subseteq N_\epsilon(\gamma)$, $d_K(X,Y) \leq \epsilon$ and $\mu(X,Y) \leq \epsilon$. In particular if Y is a leaf it is clear that $X \subseteq N_\epsilon(Y)$.

Let X and γ be as above but suppose that γ is a leaf in ∂X. Then $\overline{\mu}(X,\gamma) = d_K(X,\gamma) = 0$ so $X \subseteq N_\epsilon(\gamma)$. It follows that for leaves $\gamma \in \lambda$ which intersect with K the U_ϵ form an open cover of K.

We are now ready to describe the lamination on which we construct an earthquake approximating E. Let Λ be any finite sublamination of λ which contains Λ_0, contains only leaves of λ which intersect with K and such that the U_ϵ neighbourhoods form an open cover of K.

Lemma 2.19 *Let A be a stratum in $S(\Lambda, K)$. There exists $X \in A^*$ such that $X \subseteq N_\epsilon(\gamma)$ for all $\gamma \subseteq \partial A$. Moreover for all $Y \in A^*$*

$$d(X,Y) \leq \epsilon \text{ and } \mu(X,Y) \leq \epsilon.$$

Proof: We split the proof into three cases.

Case 1: ∂A contains only one leaf, γ. Fix any $X \in A^*$. Let Y be any other element of A^*. We have $X, Y \subseteq N_\epsilon(\gamma)$. By Lemma 2.18 either X separates Y from γ or Y separates X from γ. Either way the lemma follows from the properties of $N_\epsilon(\gamma)$.

Case 2: ∂A contains two leaves, γ and γ'. $A \cap K$ is connected and is contained in the union of two non-empty open sets $U_\epsilon(\gamma) \cap A \cap K$ and

$U_\epsilon(\gamma') \cap A \cap K$. Therefore the latter intersect and we can choose $X \in A^*$ such that $X \subseteq N_\epsilon(\gamma) \cap N_\epsilon(\gamma')$. By Lemma 2.18 any $Y \in A^*$ such that $Y \neq X$ either separates X from γ or from γ'.

Case 3: ∂A contains three or more leaves. By Lemma 2.17 A^* is the singleton set $\{X\}$ say. Since each leaf in ∂A is a leaf of ∂X we see that the first assertion of the lemma holds. The rest of the lemma is immediate. □

We continue with the proof of Theorem 2.16. Let $c : S(\Lambda, K) \to S(\lambda, K)$ be a map which chooses from each $A \in S(\Lambda, K)$ a stratum $X \in A^*$ with the properties described in Lemma 2.19. Define the metric ν on $S(\Lambda, K)$ by setting

$$\nu(A, B) = \mu(c(A), c(B)).$$

Lemma 2.17 implies that if B separates A from C then $c(B)$ separates $c(A)$ from $c(C)$. Therefore ν is a transverse metric. If H is a stratum of Λ which does not intersect with K then ν is not defined on H. Since H is a half-plane such that ∂H intersects with K we can define $\nu(H, H) = 0$ and $\nu(H, A) = \nu(\partial H, A)$ for each stratum $A \neq H$. Extended in this way ν becomes a transverse metric on all strata of Λ.

Suppose now that λ, μ is uniformly bounded with constants M, η. We show that Λ, ν is uniformly bounded. Let x, y be points of $\mathbb{H}^2$ lying in different strata of Λ. There exist points x', y' on the geodesic segment $[x, y]$ such that

1. the strata of λ which contain x', y' intersect with K and
2. $\nu(x', y') = \nu(x, y)$.

Let X, Y be the strata of λ which contain x' and y' respectively. Write $c(X)$ for $c(A)$ where A is the stratum of Λ which contains X. By Lemma 2.19 we have

$$|\mu(x', y') - \nu(x', y')| = |\mu(X, Y) - \mu(c(X), c(Y))| \leq 2\epsilon.$$

We deduce that if $d(x, y) \leq \eta$ then $\nu(x, y) \leq M + 2\epsilon$.

Fix any stratum $A \in S(\Lambda, K)$. Define F to be the unique left earthquake with source lamination Λ and shearing metric ν such that $(F|A) = (E|c(A))$. We show that F $O(\epsilon)$-approximates E on K. The precise function of ϵ depends only on the diameter of K and on an upper bound for the values of μ on K.

Fix any $X \in S(\lambda, K)$. Let B be the stratum of Λ which contains X. Let I be a geodesic which joins A to B. Let $A_1, A_2, \ldots, A_n$ be the sequence of strata of Λ through which I passes. (We have $A_1 = A$ and $A_n = B$.) This sequence is independent of I. Let K' be a set which contains K and its image under any of the maps

$$\mathrm{cmp}_F(A_i, A_{i+1}) \circ \ldots \circ \mathrm{cmp}_F(A_{n-1}, B)$$

for $i = 1, \ldots, n$. Since ν is bounded on K we may assume that K' is independent of B.

We begin with an expression for $(E|X)$, namely

$$\mathrm{cmp}_E(c(A), c(A_2)) \circ \ldots \circ \mathrm{cmp}_E(c(A_{n-1}), c(B)) \circ \mathrm{cmp}_E(c(B), X),$$

and convert it term by term, going from right to left, into the corresponding expression for $(F|B)$.

Observe that $c(B) = c(X)$ and that $\mu(c(X), X) \leq \epsilon$. By Lemma 2.1 $\mathrm{cmp}(c(B), X)$ differs from the identity by $O(\epsilon)$ on K.

For each pair A_i, A_{i+1} either A_i is a leaf in ∂A_{i+1} or vice versa. Therefore by Lemma 2.19 we have $d_K(c(A_i), c(A_{i+1})) < \epsilon$. By definition of ν we have $\nu(A_1, A_{i+1}) = \mu(c(A_1), c(A_{i+1}))$. It follows from Lemmas 2.1, 2.3 and Estimate 1 in Section 2, that $\mathrm{cmp}_E(c(A_i), c(A_{i+1}))$ differs from $\mathrm{cmp}_F(A_i, A_{i+1})$ by $O(\epsilon)\mu(c(A_i), c(A_{i+1}))$ on K'.

We have shown therefore that $(E|X)$ differs from $(F|B)$ by $O(\epsilon)$ on K. This completes the proof of Theorem 2.16.

Finally in this Section we prove a useful corollary of Theorem 2.16.

Theorem 2.20 *Let E be a uniformly bounded left earthquake. Then E is measurable and area preserving.*

Proof: Let E be uniformly bounded with constants M and η. Let f be any continuous function with compact support. Let E_n be a sequence of simple left earthquakes which converge to E uniformly on each compact subset of H^2 and which are uniformly bounded with constants $M+1$ and η. The supports of $f \circ E_n$ and of $f \circ E$ are all contained inside a single compact set which we call K. We have $f \circ E_n \to f \circ E$ uniformly on K. Since

$$\int_K f \circ E_n \, dA = \int_{\mathrm{H}^2} f \, dA$$

for all n we deduce that $f \circ E$ is integrable and that

$$\int_{\mathrm{H}^2} f \circ E \, dA = \int_{\mathrm{H}^2} f \, dA.$$

The theorem follows. $\square$

The following Lemmas will be used in the construction of an earthquake which uniformly approximates E on K. The idea of the construction is first to find an appropriate sublamination of the source lamination of E. Then to put a transverse metric on the sublamination which is derived from the shearing metric of E. The approximating earthquake will have this metrized lamination as its source lamination.

3 Quasisymmetric maps and uniformly bounded earthquakes

3.1 Quasisymmetric maps

We begin by giving a definition of quasisymmetry which is more natural than Ahlfors' definition (given in the introduction). We show that the new definition is equivalent to Ahlfors' and finally that the quasisymmetric maps of the circle form a group under composition.

We define the *cross ratio* $\mathrm{cr}(pq;rs)$ of four points $p,q,r,s \in \mathbb{C} \cup \{\infty\}$ to be

$$\mathrm{cr}(pq;rs) = \frac{p-r}{p-s} \cdot \frac{q-s}{q-r}.$$

Cross ratio is invariant under permutations of its arguments which are even and of order 2 (the Klein 4-group), and under Möbius transformations of $\mathbb{C} \cup \{\infty\}$. We shall make use of the identity

$$\mathrm{cr}(pq;rs) = 1 - \mathrm{cr}(pr;qs).$$

The cross ratio of a concyclic or collinear quadruple is real.

Definition Let f be a map of a circle in $\mathbb{C} \cup \{\infty\}$ to itself. f is K-*quasisymmetric* ($K \geq 1$) if it is a homeomorphism and satisfies the following. For every quadruple of points p,q,r,s on the circle such that $\mathrm{cr}(pq;rs) = 1/2$ we have

$$\frac{1}{K+1} \leq \mathrm{cr}(f(p)f(q);f(r)f(s)) \leq \frac{K}{K+1}.$$

This definition is clearly invariant under composition of f with Möbius transformations.

Let f be a map of $\mathbb{R} \cup \{\infty\}$ fixing ∞ which is order preserving and K-quasisymmetric according to the above definition. Let p,q,r,s be the points $x-t, \infty, x, x+t$ respectively where $x,t \in \mathbb{R}$ and $t > 0$. Since $\mathrm{cr}(pq;rs) = 1/2$ we have

$$\frac{1}{K+1} \leq \frac{f(p)-f(r)}{f(p)-f(s)} \leq \frac{K}{K+1}$$

which reduces to

$$\frac{1}{K} \leq \frac{f(x+t)-f(x)}{f(x)-f(x-t)} \leq K.$$

Therefore the cross ratio form of quasisymmetry implies Ahlfors' definition.

Let f be a K-quasisymmetric map of $\mathbb{R}$ according to Ahlfors' definition. We deduce the following. For all points $p, r, s \in \mathbb{R} \cup \{\infty\}$ such that $p < r < s$ and for each integer $n \geq 1$

$$\frac{p-r}{p-s} \leq \frac{1}{2^n} \;\Rightarrow\; \frac{f(p)-f(r)}{f(p)-f(s)} \leq \left(\frac{K}{K+1}\right)^n,$$
$$\frac{p-r}{p-s} \geq \frac{1}{2^n} \;\Rightarrow\; \frac{f(p)-f(r)}{f(p)-f(s)} \geq \frac{1}{(K+1)^n}.$$

Lemma 3.1 *Let f be a K-quasisymmetric map of $\mathbb{R}$ according to Ahlfors' definition. Let p, q, r, s be a quadruple of points in $\mathbb{R} \cup \{\infty\}$ such that $\mathrm{cr}(pq; rs) \in (0,1)$. Let $n \geq 1$ be an integer such that*

$$\frac{1}{2^n} \leq \mathrm{cr}(pq; rs) \leq 1 - \frac{1}{2^n}.$$

Then

$$\frac{1}{(K+1)^{n+1}} < \mathrm{cr}(f(p)f(q); f(r)f(s)) < 1 - \frac{1}{(K+1)^{n+1}}.$$

Proof: Since cross ratios are invariant under the even permutations of order two, we may assume that p is finite and that the order of the points is either $p < q < s < r$ or $p < r < s < q$. (We are making use of the fact that the cross ratio lies in the range $(0,1)$.) We begin by proving the left hand inequality.

Case 1, $p < q < s < r$.
At least one of

$$\frac{p-r}{p-s} \quad \text{or} \quad \frac{p-r}{q-r}$$

is less than 2. Without loss of generality we suppose it is the former. Then since $\mathrm{cr}(pq; rs) \geq 1/2^n$ we have

$$\frac{q-s}{q-r} > \frac{1}{2^{n+1}}.$$

Quasisymmetry implies

$$\frac{f(q)-f(s)}{f(q)-f(r)} > \frac{1}{(K+1)^{n+1}}.$$

Since $\dfrac{f(p)-f(r)}{f(p)-f(s)} > 1$, we obtain

$$\mathrm{cr}(f(p)f(q); f(r)f(s)) > \frac{1}{(K+1)^{n+1}}.$$

Case 2, $p < r < s < q$.
There exist integers $n_1, n_2 \geq 1$ such that

$$\frac{p-r}{p-s} \geq \frac{1}{2^{n_1}} \quad , \quad \frac{q-s}{q-r} \geq \frac{1}{2^{n_2}}$$

and $n_1 + n_2 = n+1$. At least one of the above inequalities is strict. Quasisymmetry implies

$$\frac{f(p)-f(r)}{f(p)-f(s)} \geq \frac{1}{(K+1)^{n_1}} \quad , \quad \frac{f(q)-f(s)}{f(q)-f(r)} \geq \frac{1}{(K+1)^{n_2}}$$

where at least one of the above inequalities is strict. Therefore

$$\mathrm{cr}(f(p)f(q); f(r)f(s)) > \frac{1}{(K+1)^{n+1}}.$$

This proves the left hand inequality in the statement of Lemma 3.1. By applying the result to $\mathrm{cr}(pr; qs)$ instead we obtain

$$1 - \mathrm{cr}(f(p)f(q); f(r)f(s)) > \frac{1}{(K+1)^{n+1}}.$$

This completes the proof of Lemma 3.1. □

Lemma 3.1 with $n = 1$ says that Ahlfors' definition of quasisymmetry with constant K implies the cross ratio definition with constant $K^2 + 2K$.

We need the following lemma in order to show that the quasisymmetric maps of a fixed circle form a group under composition.

Lemma 3.2 *Let f and p, q, r, s be as in Lemma 3.1. Suppose we have an integer $n \geq 1$ such that*

$$0 < \mathrm{cr}(pq; rs) \leq \frac{1}{2^n}.$$

Then

$$0 < \mathrm{cr}(f(p)f(q); f(r)f(s)) < 2\left(\frac{K}{K+1}\right)^n.$$

Proof: As before it is sufficient to consider the two cases $p < q < s < r$ and $p < r < s < q$.

Case 1, $p < q < s < r$.
At least one of

$$\frac{f(p)-f(r)}{f(p)-f(s)} \quad \text{and} \quad \frac{f(p)-f(r)}{f(q)-f(r)}$$

is less than 2. Without loss of generality suppose it is the former. Since $\dfrac{p-r}{p-s} > 1$ we must have

$$\frac{q-s}{q-r} < \frac{1}{2^n}.$$

Quasisymmetry implies

$$\frac{f(q)-f(s)}{f(q)-f(r)} < \left(\frac{K}{K+1}\right)^n$$

and therefore

$$\mathrm{cr}(f(p)f(q); f(r)f(s)) < 2\left(\frac{K}{K+1}\right)^n.$$

Case 2, $p < r < s < q$
By an argument rather similar to the one given in Case 2 in the proof of Lemma 3.1 we prove that

$$\mathrm{cr}(f(p)f(q); f(r)f(s)) < \left(\frac{K}{K+1}\right)^{n-1}.$$

Since $\dfrac{K+1}{K} \leq 2$ this implies the desired result. □

By switching r and q in the statement of Lemma 3.2 we obtain a similar result for cross ratios close to 1.

Let f be K-quasisymmetric and g K'-quasisymmetric (cross-ratio definition). Pick an integer n such that

$$2\left(\frac{K'}{K'+1}\right)^n \leq \frac{1}{K+1}.$$

By applying $g^{-1} \circ f$ to quadruples with cross-ratio 1/2 and using Lemma 3.2 we deduce that $g^{-1} \circ f$ is $(2^n - 1)$-quasisymmetric. It follows that the quasisymmetric maps form a group under composition.

3.2 Cross ratios and the thrice punctured sphere

In the previous section we gave bounds on how the cross ratio of a quadruple of distinct points could vary under the application of a quasisymmetric map. The range of the cross ratio function is a hyperbolic surface, namely $\tilde{\mathbb{C}} - \{0, 1, \infty\}$ with its Poincaré metric. We show in this Section that the cross ratio of any quadruple of points on the circle is moved a bounded hyperbolic distance under the application of a fixed quasisymmetric map.

We denote by ρ the Poincaré metric on $\tilde{\mathbb{C}} - \{0, 1, \infty\}$. The following well known lemma, which is a corollary of the Schwarz Lemma, enables us to find bounds on ρ.

Lemma 3.3 *Let F and G be complete connected hyperbolic surfaces and let h be a conformal map of F into G. Then h is either a local isometry or a strict contraction.*

Lemma 3.4 *Let ρ_0 be the metric on $(0,1)$ defined by*

$$\rho_0(x, y) = |\log(-\log x) - \log(-\log y)|$$

for all $x, y \in (0,1)$. Then $\rho(x, y) \leq \rho_0(x, y)$ for all $x, y \in (0,1)$.

Proof: Let $\tilde{\rho}$ be the Poincaré metric on $\mathbb{D} - \{0\}$. Observe that the map

$$z \mapsto \exp(iz)$$

makes the upper half plane into the universal cover of $\mathbb{D} - \{0\}$. From this we see that ρ_0 is simply the restriction of $\tilde{\rho}$ to $(0, 1)$.

By Lemma 3.3, the inclusion map of $\mathbb{D} - \{0\}$, with metric $\tilde{\rho}$, into $\tilde{\mathbb{C}} - \{0, 1, \infty\}$, with metric ρ, is distance decreasing. The lemma follows. □

Note: In fact as we approach 0 the ratio of $\tilde{\rho}$ to ρ tends to 1. We obtain a similar approximation to ρ near 1 by taking the Poincaré metric on $D_1(1) - \{1\}$. We do not prove these statements since no actual use will be made of them.

Theorem 3.5 *Let f be a K-quasisymmetric map (cross ratio definition) of a circle S. Let ρ be the Poincaré metric on $\mathbb{D} - \{0\}$. Then there exists a constant $M = M(K)$ such that*

$$\rho(\mathrm{cr}(pq; rs), \mathrm{cr}(f(p)f(q); f(r)f(s))) \leq M$$

for all quadruples (p, q, r, s) of distinct points on S.

Proof: Since the cross ratio of a concyclic quadruple is real we know that

$$cr(pq,rs) \in \mathbb{R} - \{0,1\}.$$

Permuting the arguments of the cross ratio function has the same effect as composing it with one of the six possible orientation preserving isometries of $\hat{\mathbb{C}} - \{0,1,\infty\}$. Therefore we may assume without loss of generality that

$$0 < cr(pq;rs) \leq \frac{1}{2}.$$

Let us now write cr for $cr(pq;rs)$ and cr$'$ for $cr(f(p)f(q);f(r)f(s))$. Let n_0 be the least integer such that

$$n_0 \log\left(\frac{K+1}{K}\right) \geq 2\log 2.$$

Notice in particular that $n_0 \geq 2$.

We consider two possible cases, depending on the value of cr.

If we have

$$\frac{1}{2^{n_0}} \leq \mathrm{cr} \leq \frac{1}{2}$$

then it follows from Lemma 3.1 that

$$\frac{1}{(K+1)^{n_0+1}} \leq \mathrm{cr}' \leq 1 - \frac{1}{(K+1)^2}.$$

In this case it is straightforward to write down an upper bound for $\rho_0(\mathrm{cr},\mathrm{cr}')$.

If the first case does not hold then there exists an $n \geq n_0$ such that

$$\frac{1}{2^{n+1}} \leq \mathrm{cr} \leq \frac{1}{2^n}.$$

Then it follows from Lemmas 3.1 and 3.2 that

$$\frac{1}{(K+1)^{n+2}} \leq \mathrm{cr}' \leq 2\left(\frac{K}{K+1}\right)^n.$$

Using the definition of ρ_0, the inequalities satisfied by n_0 and the fact that $n \geq n_0$, a straightforward calculation gives

$$\rho_0(\mathrm{cr},\mathrm{cr}') \leq \max\left\{\log\left(\frac{2\log(K+1)}{\log 2}\right), \log\left(\frac{4\log 2}{\log(1+K^{-1})}\right)\right\}.$$

Applying Lemma 3.4 completes the proof of this theorem. □

3.3 Uniformly bounded earthquakes

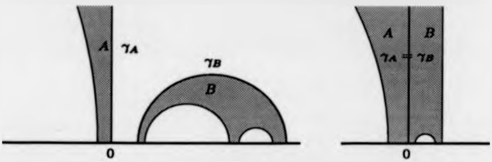
We show in this section that an earthquake is uniformly bounded if and only if it gives rise to a quasisymmetric map of the circle at infinity. This provides an important link between the theory of earthquakes and the complex analytic view of Teichmüller space.

Throughout this section we use the cross ratio definition of quasisymmetry.

Theorem 3.6 *Let E be a left earthquake with quasisymmetric boundary values. Then E is uniformly bounded with constants depending on the constant of quasisymmetry.*

Proof: Let A and B be strata of λ. If A is a gap then let γ_A be the leaf in ∂A which weakly separates A from B. If A is a leaf then let $\gamma_A = A$. Define γ_B similarly.

Choose coordinates in the upper half plane for the domain and range of E as follows. Let γ_A lie along the positive imaginary axis and let B lie to the right of A. We obtain a similar picture in the range by requiring that $(E|A)$ be the identity. If γ_A and γ_B share exactly one common endpoint then by switching A and B if necessary we may assume that this is at ∞.



Shown here are two examples in which coordinates have been chosen as described above.

Let K be the constant of quasisymmetry which applies to the boundary mapping of E. Let x be a point on the positive real axis. Applying the definition of K-quasisymmetry to the quadruple $(-x, \infty, 0, x)$ we find $E(x) \leq Kx$. The comparison isometry $\mathrm{cmp}(A, B)$ is simply $(E|B)$. Because E is a left earthquake $(E|B)(x) \leq E(x)$ for all x weakly separated from A by B.

We consider the two cases $\gamma_A = \gamma_B$ and $\gamma_A \neq \gamma_B$.

Case 1, $\gamma_A = \gamma_B$. Since the axis of $\mathrm{cmp}(A, B)$ is the positive imaginary axis $\mathrm{cmp}(A, B)$ is simply an enlargement. The above inequalities imply $|\mathrm{cmp}(A, B)| \leq \log K$.

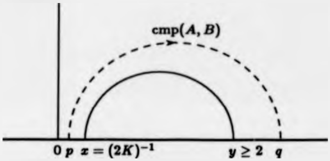
Case 2, $\gamma_A \neq \gamma_B$. Label the endpoints of γ_B as x and y where $0 < x < y \leq \infty$. The distance $d(A, B)$, equal to $d(\gamma_A, \gamma_B)$, is given by the formula

$$\cosh d(A, B) = \frac{x + y}{y - x}.$$

Set

$$\eta = \cosh^{-1} \frac{4K + 1}{4K - 1}.$$

Suppose, in order to show that E is uniformly bounded, that $d(A, B) \leq \eta$. Then $y/x \geq 4K$. Conjugating E by an appropriate enlargement we may assume that $x = (2K)^{-1}$ and therefore $y \geq 2$. Let p be the repelling endpoint of the axis of $(E|B)$ and q the attracting endpoint. The situation is now as shown below.



For each $t \in \mathbb{R}$ not equal to p or q we have

$$|(E|B)| = \log(\mathrm{cr}(pq; (E|B)(t)t)).$$

Set $t = 1/K$. We have the following inequalities:

$$0 < p \leq \frac{1}{2K}, \quad 2 \leq q \leq \infty, \quad (E|B)(t) \leq 1$$

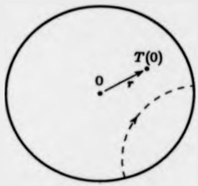
where the last of these follows from quasisymmetry. Maximizing the cross ratio subject to these inequalities, and using the fact that $(E|B) = \mathrm{cmp}(A, B)$, we find that

$$|\mathrm{cmp}(A, B)| \leq \log(4K - 3).$$

We have shown therefore that E is uniformly bounded with constants η as above and $M = \log(4K - 3)$. $\square$

We turn next to proving the converse of Theorem 3.6. To prepare the way we prove first a number of lemmas.

Lemma 3.7 *Let T be an isometry mapping the Poincaré disk $\mathbb{D}$ to itself. Let r be the hyperbolic distance from 0 to $T(0)$. Then $T|_{\partial\mathbb{D}}$ is bi-Lipschitz with constant e^r.*



Proof: We can express T as a fractional linear transformation of the form

$$T(z) = \beta . \frac{z - \alpha}{1 - \overline{\alpha} z}$$

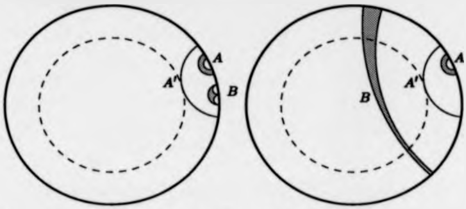
where $|\beta| = 1$ and $|\alpha| < 1$. We can assume, without loss of generality, that $\beta = 1$. Write T' for the derivative of T as a holomorphic function. $|T'|$ takes its maximum value at $\alpha/|\alpha|$ and its minimum at $-\alpha/|\alpha|$. At $\alpha/|\alpha|$ its value is e^r and at $-\alpha/|\alpha|$ it is e^{-r}. $\square$

Let λ be a geodesic lamination on $\mathbb{H}^2$. Let X and Y be strata of λ which intersect with a geodesic segment $[a, b]$. We say that Y is *further from a than* X if X weakly separates Y from a.

Let K be a compact convex subset of $\mathbb{H}^2$ and let o be a point on K. Let A be a stratum of λ and suppose that A does not intersect with K. Let a be a point on A. We say that A' *encloses* A if, of the strata which intersect with both $[o, a]$ and K, A' is the furthest from o.

Consider the set of leaves which intersect with both $[o, a]$ and K. Let γ be the one which maximises hyperbolic distance from o. There can be at most one stratum, intersecting with both K and $[o, a]$, further from o than γ. Therefore there is a unique stratum A' enclosing A.

Let B be another stratum of λ. This diagram shows two positions in which B could possibly turn up.



They correspond to the two cases listed in the following lemma.

Lemma 3.8 *Let A and B be strata of λ and suppose that A is disjoint from K. Let A' be the stratum enclosing A. Either*

1. *A' weakly separates A from B or*
2. *B is disjoint from K and A' also encloses B.*

Proof: Let x be a point on A and y a point on B. By Lemma 2.5 there exists a stratum C which intersects with all three edges of the triangle (o, x, y).

If C intersects with K then either $A' = C$ or A' is further from o than C. Either way A' must intersect with $[x, y]$ and therefore case 1 in the statement of the lemma holds.

If C does not intersect with K then B cannot intersect with K either. Let B' denote the stratum enclosing B. C' is further from o than either A' or B'. Therefore A' and B' both intersect with $[o, x]$ and $[o, y]$. It follows that each is further from o than the other and therefore they are the same. $\square$

Let K and λ be as above. A denotes a stratum of λ. If A does not intersect with K, we write A' for the stratum of λ which encloses A. If A intersects with K, we set $A' = A$.

Let E be a λ-left earthquake on $\mathbf{H}^2$ which fixes the stratum of λ containing o. Define maps E_1 and E_2 as follows. Set

$$\begin{aligned} E_1|_A &= \mathrm{cmp}_E(A', A)|_A \text{ and} \\ E_2|_A &= (E|A')|_A \end{aligned}$$

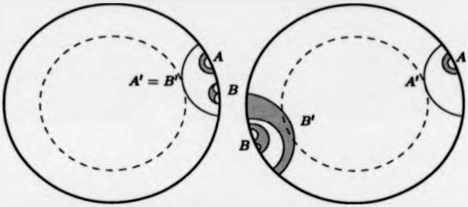
for all strata A of λ.

Lemma 3.9 *E_1 and E_2 are λ-left earthquakes and satisfy $E = E_2 \circ E_1$.*

Proof: It is clear from the definition that E_i agrees with an isometry on each stratum of λ. Let A and B be strata of λ. Lemma 3.8 implies that either

1. $A' = B'$ or
2. a geodesic segment running from A to B crosses first A' then B'.

(We do not exclude from the second case the possibility that $A = A'$ or $B = B'$.) A typical example of each of these cases is shown here.



It is now straightforward to check that $\mathrm{cmp}_{E_i}(A, B)$, whenever non-trivial, is a translation with axis weakly separating A from B.

We check now that $E = E_2 \circ E_1$. Let A be any stratum of λ. If A intersects with K then E_1 restricts to the identity and E_2 agrees with E on A.

Suppose A does not intersect with K. Lemma 3.8 implies that each stratum, lying in the same component of the complement of A' as A, is enclosed by A'. In particular each stratum which intersects with $E_1(A)$ is enclosed by A'. Therefore E_2 agrees with $(E|A')$ on the whole of $E_1(A)$. It follows that $E_2 \circ E_1$ agrees with E on A. □

Lemma 3.10 *Let E be a uniformly bounded left earthquake. Let E_2 be defined as above. Then $E_2|_{\partial\mathbb{D}}$ is bi-Lipschitz, with constant depending only on K and the constants of uniform boundedness for E.*

Proof: Write λ_2 for the source lamination of E_2. Let X be any stratum of λ_2. In view of the definition of E_2 there exists a stratum A of λ, intersecting with K, such that $(E_2|X) = (E|A)$. Uniform boundedness of E and Estimate 1 of Section 2.2 give an upper bound for $\|(E|A)\|$ and thus for $\|(E_2|X)\|$. From Lemmas 2.1 and 3.7 we can derive a bound M' on the bi-Lipschitz constant of $(E_2|X)|_{\partial\mathbb{D}}$.

Let x and y be points on $\partial\mathbb{D}$. Write $d(x,y)$ for the shortest Euclidean distance from x to y, measured along $\partial\mathbb{D}$. We show that

$$d(x,y)/M' \leq d(E_2(x), E_2(y)) \leq M'd(x,y).$$

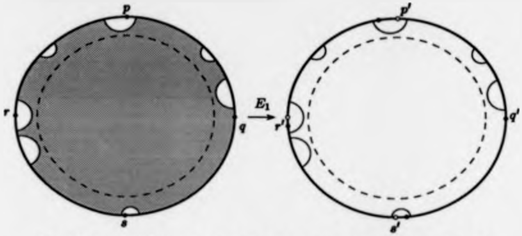
Let X and Y be strata of λ_2 which contain x and y respectively in their ideal boundaries. Define a map f, from $\partial\mathbb{D}$ to itself, as follows. If $\mathrm{cmp}_{E_2}(X,Y)$ is non-trivial then the endpoints of its axis divide $\partial\mathbb{D}$ into two intervals. Set f equal to $(E_2|X)$ on the interval containing x, and to $(E_2|Y)$ on the interval containing y. If cmp_{E_2} is trivial, so that $(E_2|X) = (E_2|Y)$, simply set $f = (E_2|X)|_{\partial\mathbb{D}}$. So defined, f is continuous, M'-bi-Lipschitz, and satisfies $f(x) = E_2(x)$ and $f(y) = E_2(y)$. Applying the definition of M'-bi-Lipschitz to f for x and y, we deduce the above inequalities. Since x and y were arbitrary, the lemma follows. □

Theorem 3.11 *Let E be a uniformly bounded left earthquake. Then the boundary mapping of E is quasisymmetric with constant depending only on the constants of uniform boundedness for E.*

Proof: Let (p,q,r,s) be a quadruple of points on the ideal boundary of the domain of E such that $\mathrm{cr}(p,q;r,s) = 1/2$. Choose coordinates in $\mathbb{D}$ for the domain of E so that (p,q,r,s) are $(i,1,-1,-i)$ respectively. Choose coordinates in $\mathbb{D}$ for the range of E such that E fixes the stratum containing 0.

Let $\epsilon = 0.1$. Let K be the closed disk of Euclidean radius $1-\epsilon$ and centre 0. Define E_1 and E_2 as above.

We show that $|z - E_1(z)| < 1/4$ for each complex number $z \in \partial\mathbb{D}$. Write λ_1 for the source lamination of E_1. No leaf of λ_1 intersects with the interior of K. Write O for the stratum of λ_1 which contains 0. An easy calculation shows that the Euclidean diameter of each component of $\overline{\mathbb{D}} - \overline{O}$ is bounded by $\epsilon\left(\frac{2-\epsilon}{1-\epsilon}\right)$. For $\epsilon = 0.1$ this is less than $1/4$. Since E_1 fixes $\partial\mathrm{D} \cap \overline{\mathcal{O}}$ pointwise and maps each component of $\partial\mathrm{D} - \overline{\mathcal{O}}$ into itself, E_1 moves no point further than the specified distance.

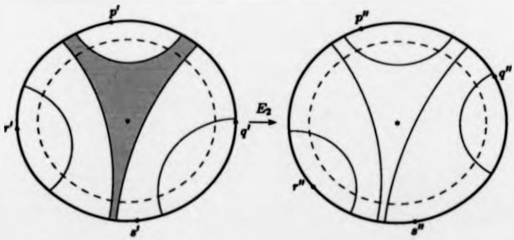


Let p' denote $E_1(p)$ etc. From the above estimate it is clear that $|p' - r'|, |r' - s'|, |s' - q'|$ and $|q' - p'|$ all exceed $(\sqrt{2} - 1/2)$.

Now we turn our attention to E_2. By Lemma 3.10 $E_2|_{\partial\mathrm{D}}$ is M'-bi-Lipschitz where M' depends only on M and η. It follows that

$$|E_2(x) - E_2(y)| \geq |x - y|/M',$$

for all $x, y \in \partial\mathrm{D}$.



Finally let $p'' = E_2(p') = E(p)$ etc. From the above estimates we see that $|p'' - r''|, |r'' - s''|, |s'' - q''|$ and $|q'' - p''|$ all exceed $(\sqrt{2} - 1/2)/M'$. Therefore $\mathrm{cr}(p'', q''; r'', s'')$ is bounded away from 0 and 1. □

4 An approximation theorem

The remainder of this paper is devoted to proving that we can approximate a uniformly bounded earthquake by a bi-Lipschitz diffeomorphism in a fairly natural way. (This will be made precise later.) This gives a direct way of seeing how an earthquake relates two hyperbolic structures. It also provides a second link between earthquakes and the view of Teichmüller space as a space of conformal structures on a surface.

4.1 Some preparatory definitions and analysis

In this section we set up the notation which we use in Section 4.3 and show that certain objects we define are suitably 'well behaved.'

Fix the following notation throughout this section. Let $\mathbb{D}_1$ and $\mathbb{D}_2$ be two copies of $\mathbb{H}^2$. Let (λ, μ) be a uniformly bounded metrized lamination on $\mathbb{D}_1$. Let $E_t : \mathbb{D}_1 \to \mathbb{D}_2$, for $t \in \mathbb{R}^+$, be a 1-parameter family of surjective λ-left earthquakes such that E_t has shearing metric $t\mu$. It follows from Theorem 2.6 and Theorem 2.10 that such families exist.

We denote by μ_t the transverse metric on $E_t(\lambda)$ defined by

$$\mu_t(A, B) = \mu(E_t^{-1}A, E_t^{-1}B).$$

Lemma 4.1 *Let μ be uniformly bounded with constants η and M. For $t \in \mathbb{R}^+$ the metric μ_t is uniformly bounded with constants ηe^{-2tM} and M.*

Proof: Fix $t \in \mathbb{R}^+$. We prove first that for all strata A, B of λ,

$$d(A, B) > \eta \quad \Rightarrow \quad d(E_tA, E_tB) > \eta e^{-2tM}.$$

Let A and B be strata which satisfy $d(A, B) > \eta$. We consider two cases.

Case 1: $\mu(A, B) < 2M$. Lemma 2.8 implies $d(E_tA, E_tB) > \eta e^{-2tM}$.

Case 2: $\mu(A, B) \geq 2M$. Since μ has no discontinuity of size greater than M we can find a stratum C which weakly separates A from B and satisfies $M < \mu(A, C) \leq 2M$. The uniform bounds on μ imply $d(A, C) > \eta$. Lemma 2.8 implies $d(E_tA, E_tC) > \eta e^{-2tM}$ which forces the required result.

Now let A' and B' be strata in $E_t(\lambda)$. Using the inequality just proved, we deduce

$$\begin{aligned} d(A',B') \leq \eta e^{-2tM} &\Rightarrow d(E_t^{-1}A', E_t^{-1}B') \leq \eta \\ &\Rightarrow \mu(E_t^{-1}A', E_t^{-1}B') \leq M \\ &\Rightarrow \mu_t(A',B') \leq M. \end{aligned}$$

This completes the proof of Lemma 4.1. □

Lemma 4.2 *For each $s,t \in \mathbb{R}^+$ the map $E_{s+t} \circ E_t^{-1}$ is a left earthquake with source lamination $E_t(\lambda)$ and shearing metric $s\mu_t$.*

Proof: $E_{s+t} \circ E_t^{-1}$ restricts to an isometry on each stratum of $E_t(\lambda)$. We prove that the comparison isometries of this map have weakly separating axes and that it has shearing metric $s\mu_t$. Let x and y be points in $\mathbb{D}_1$ and let $P = \{x_0, \ldots, x_n\}$ be a partition of $[x,y]$. Let $T_{P,t} = T_{1,t} \circ \ldots \circ T_{n,t}$ be the P-compatible isometry defined in Example 1 of Section 2 with reference to the measured lamination $(\lambda, t\mu)$. In accordance with the notation used in that example we denote by γ_i the axis of $T_{i,t}$ whenever the latter is non-trivial.

We have

$$T_{P,s+t} \circ T_{P,t}^{-1} = T_{1,s+t} \circ \ldots \circ T_{n,s+t} \circ T_{n,t}^{-1} \circ \ldots \circ T_{1,t}^{-1}.$$

When $\mu(x_{i-1}, x_i) \neq 0$ the isometry $T_{i,s+t} \circ T_{i,t}^{-1}$ is a distance $s\mu(x_{i-1}, x_i)$ translation along γ_i. Define

$$S_{i,s} = (T_{1,t} \circ \ldots \circ T_{i-1,t}) \circ T_{i,s} \circ (T_{1,t} \circ \ldots \circ T_{i-1,t})^{-1}.$$

When $\mu(x_{i-1}, x_i) \neq 0$ the isometry $S_{i,s}$ is a distance $s\mu(x_{i-1}, x_i)$ translation along $T_{1,t} \circ \ldots \circ T_{i-1,t}(\gamma_i)$. The above equations imply that

$$T_{P,s+t} \circ T_{P,t}^{-1} = S_{1,s} \circ \ldots \circ S_{n,s}.$$

The axes of the $S_{i,s}$'s are disjoint and weakly separate x from $T_{P,t}(y)$. Lemma 2.2 implies that the axis of $T_{P,s+t} \circ T_{P,t}^{-1}$ weakly separates x from $T_{P,t}(y)$ and that the following holds.

$$s\mu(x,y) \leq |T_{P,s+t} \circ T_{P,t}^{-1}| \leq s\mu(x,y) \cosh d(x, T_{P,t}(y)).$$

Take limits as P is refined. We deduce that the axis of $\mathrm{cmp}_{E_{s+t}}(x,y) \circ \mathrm{cmp}_{E_t}(x,y)^{-1}$ weakly separates x from $T_{P,t}(y)$ and

$$s\mu(x,y) \leq |\mathrm{cmp}_{E_{s+t}}(x,y) \circ \mathrm{cmp}_{E_t}(x,y)^{-1}| \leq s\mu(x,y) \cosh d(x, \mathrm{cmp}_{E_t}(x,y)(y)).$$

Let $x' = E_t(x)$ and $y' = E_t(y)$. We have

$$\mathrm{cmp}_{E_{s+t}}(x,y) \circ \mathrm{cmp}_{E_t}(x,y)^{-1} = (E_t|x)^{-1} \circ \mathrm{cmp}_{E_{s+t} \circ E_t^{-1}}(x',y') \circ (E_t|x).$$

Therefore the axis of $\mathrm{cmp}_{E_{s+t} \circ E_t^{-1}}(x',y')$ weakly separates x' from y'. The above inequalities can be rewritten

$$s\mu_t[x',y'] \leq |\mathrm{cmp}_{E_{s+t} \circ E_t^{-1}}(x',y')| \leq s\mu_t[x',y'] \cosh d(x',y')$$

which is sufficient to ensure that $E_{s+t} \circ E_t^{-1}$ has shearing metric $s\mu_t$. □

Notation: Let $\Psi_t : \mathbb{D}_1 \to \mathbb{D}_2$ be a 1-parameter family of bijections which is smooth in t. We denote by Ψ_t' the (possibly discontinuous) vector field on $\mathbb{D}_2$ defined by

$$\Psi_t' = \left.\frac{d}{ds}\right|_0 \Psi_{s+t} \circ \Psi_t^{-1}.$$

Using the Poincaré disk model for hyperbolic space we identify $\mathbb{D}_2$ with the open unit disk $\mathbb{D} \subset \mathbb{C}$. Then we can regard Ψ_t and all of its derivatives with respect to t, as complex valued functions on $\mathbb{D}_1$.

Let u and v be points on $\partial\mathbb{D}$. Denote by $A(u,v)$ the function corresponding to the vector field on $\overline{\mathbb{D}}$ which is generated by translating at hyperbolic unit speed along the geodesic joining u to v.

We show next that for each fixed pair of points $x, y \in \mathbb{H}^2$, $\mathrm{cmp}_{E_t}(x,y)$ is a smooth function of t. As in the proof of Lemma 4.2 let x and y be points in $\mathbb{D}_1$ and let $P = \{x_0, \ldots, x_n\}$ be a partition of $[x,y]$. Let $T_{P,t} = T_{1,t} \circ \ldots \circ T_{n,t}$ be the P-compatible isometry defined in Example 1 of Section 2 with reference to the measured lamination $(\lambda, t\mu)$.

We have $T_{P,s+t} \circ T_{P,t}^{-1} = S_{1,s} \circ \ldots \circ S_{n,s}$ where $S_{i,s}$ is a distance $s\mu(x_{i-1}, x_i)$ translation with axis $T_{1,t} \circ \ldots \circ T_{i,t}(\gamma_i)$. Write $u_i(t)$ for the repelling fixed point of $S_{i,s}$ and $v_i(t)$ for the attracting fixed point. Using the notation explained above we have

$$T_{P,t}' = \sum_{i=0}^{n} \mu(x_{i-1}, x_i) A(u_i(t), v_i(t)). \tag{3}$$

Without loss of generality we assume that x lies at the centre of $\mathbb{D}$. For $t \in [0,1]$ the geodesics $T_{1,t} \circ \ldots \circ T_{i,t}(\gamma_i)$ lie at most a distance $\mu(x,y) + d(x,y)$ from the centre of $\mathbb{D}$. Therefore the pairs $(u_i(t), v_i(t))$ lie in a compact subset, K say, of $\partial\mathbb{D} \times \partial\mathbb{D} - \mathrm{diag}(\partial\mathbb{D})$.

Lemma 4.3 *For each integer $r \geq 1$ there is a constant C_r such that $|T_{P,t}^{(r)}| \leq C_r$. Moreover C_r is independent of $t \in [0,1]$ and the choice of partition P.*

Proof: We prove this by induction on r. Since K is compact and independent of t and P, Equation 3 implies that C_1 exists.

Suppose we have bounds $C_1, \ldots, C_r$ on the first r derivatives of $T_{P,t}$.

Let $P_i = \{x_0, \ldots, x_i\}$ for $i \leq n$. Then $u_i(t) = T_{P_i,t}(u_i(0))$. Therefore for $k \leq r$ we have $|u_i^{(k)}(t)| \leq C_k$. The derivatives up to degree r of $v_i(t)$ satisfy the same bounds.

Differentiating Equation 3 a total of r times we obtain an expression for $T_{P,t}^{(r+1)}$ as a polynomial in the partial derivatives up to degree r of A and the derivatives up to degree r of $u_i(t)$ and $v_i(t)$. The partial derivatives up to degree r of A are bounded because K is compact. We have shown that the derivatives up to degree r of $u_i(t)$ and $v_i(t)$ are bounded. Therefore $|T_{P,t}^{(r+1)}|$ is bounded and the lemma follows by induction. □

Fix a point $z \in \mathbb{D}$. For each partition P of $[x, y]$ we have the path $T_{P,t}(z)$ in $\mathbb{D}$. As P is refined, $T_{P,t}(z)$ converges uniformly to $\mathrm{cmp}_{E_t}(x, y)(z)$. Since the derivatives of $T_{P,t}(z)$ are bounded uniformly with respect to t and P it follows that $\mathrm{cmp}_{E_t}(x, y)(z)$ is smooth in t. We can also deduce that as P is refined

$$\sum_{i=0}^{n} \mu(x_{i-1}, x_i) A(u_i(t), v_i(t)) \;\; \to \;\; \mathrm{cmp}_{E_t}(x, y)'. \tag{4}$$

We assume for the remainder of this section that E_t is smooth in t.

The vector field generated by a differentiable 1-parameter family of isometries is called a Killing field. The Killing fields on $\mathbb{D}_2$ form a 3-dimensional vector space which can be identified with the Lie algebra of $\mathrm{Isom}^+(\mathbb{D}_2)$.

For each $p \in \mathbb{D}_2$ and $t \in [0, 1]$ define $W_{p,t}$ to be the Killing vector field on $\mathbb{D}_2$ which agrees with E_t' on the stratum of $E_t(\lambda)$ containing p.

Next we prove certain bounds on the W's.

Write $F_{s,t}$ for the $E_t(\lambda)$-left earthquake $E_{s+t} \circ E_t^{-1}$ where $s, t \in \mathbb{R}^+$. Recall that $F_{s,t}$ has shearing metric $s\mu_t$. For $t \in [0, 1]$ we assume that μ_t is uniformly bounded with constants η and M. (See Lemma 4.1.)

For $p, q \in \mathbb{D}_2$, by definition $(F_{s,t}|\hat{q}) = (F_{s,t}|\hat{p}) \circ \mathrm{cmp}_{F_{s,t}}(\hat{p}, \hat{q})$. Differentiating we obtain,

$$W_{q,t} = W_{p,t} + \left.\frac{d}{ds}\right|_0 \mathrm{cmp}_{F_{s,t}}(p, q).$$

Lemma 2.2 implies

$$s\mu_t[p, q] \leq |\mathrm{cmp}_{F_{s,t}}(p, q)| \leq s\mu_t[p, q] \cosh d(p, q).$$

Let x be a point in D_2 and set $r = \max\{d(x,p), d(x,q)\}$. Then

$$\mu_t[p,q] \leq d(x, \text{cmp}_{P_{s,t}}(\hat{p},\hat{q})(x))/s \leq \mu_t[p,q] \cosh d(p,q) \cosh r.$$

Letting $s \to 0$ and substituting in the previous expression, we have

$$\mu_t[p,q] \leq \|W_{p,t}(x) - W_{q,t}(x)\| \leq \mu_t[p,q] \cosh d(p,q) \cosh r. \tag{5}$$

Let $x \in D_2$ and $t \in [0,1]$ be fixed. It follows from the right-hand inequality above that, as a function of p, $W_{p,t}(x)$ is bounded on bounded subsets of D_2 and continuous off $E_t(\lambda_d)$ where λ_d, defined in Section 2.4, is the countable set of leaves on which μ is discontinuous. Since $E_t(\lambda_d)$ has measure zero it follows that $W_{p,t}(x)$ is locally integrable with respect to p.

Fix $\delta > 0$. Let ϕ be a smooth circularly symmetric bump function on D_2 with integral 1 and support contained in a disk of radius δ. Let ϕ_y be the bump function centered at y which we obtain by composing ϕ with an appropriate isometry of D_2. For each isometry T of D_2 we have

$$\phi_y \circ T = \phi_{T^{-1}y}.$$

Define

$$V_{y,t} = \int_{D_2} \phi_y(x) W_{x,t} dA(x).$$

For each $y \in D_2$ and $t \in [0,1]$, $V_{y,t}$ is a Killing field on D_2. We call $V_{y,t}$ the *approximating vector field* for E_t.

We show next that $V_{y,t}$ is smooth in (y,t). Since, by Theorem 2.20, E_t is area preserving we have

$$V_{y,t} = \int_{D_1} (\phi_y \circ E_t)(x)(E_t|x)' dA(x).$$

Notice that the integrand here depends smoothly on the pair (y,t). Let K be any bounded subset of D_2. Each partial derivative (w.r.t. y and/or t) of $\phi_y \circ E_t(x)$ is uniformly bounded w.r.t. $x \in D_1$ and $(y,t) \in K \times [0,1]$.

As we vary $(y,t) \in K \times [0,1]$ the support of $\phi_y \circ E_t$ remains inside a bounded subset of D_1 which we will call K'. Each partial derivative (w.r.t. t) of $(E_t|x)'$ is uniformly bounded w.r.t. $x \in K'$ and $t \in [0,1]$. It follows by differentiating under the integral sign that $V_{y,t}$ is smooth in (y,t).

4.2 Killing fields, geodesics and the distance function

We discuss properties of Killing fields, geodesics and the distance function which will be useful in Section 4.3.

Let X be a vector field on $\mathbb{H}^2$ and let γ be an oriented geodesic. Let p be a point on γ. Let $U(p)$ be the unit tangent vector along γ at p which defines the orientation of γ. We define $< X(p), \gamma >$ *the component of* X *along* γ *at* p to be the real number $< X(p), U(p) >$, where $< .,. >$ indicates the inner product associated with the hyperbolic metric.

Let w be a point of $\mathbb{H}^2$. The function $d(., w)$, assigning to each point of $\mathbb{H}^2$ its distance from w, is smooth on $\mathbb{H}^2 - \{w\}$. The derivative of this function is a 1-form on $\mathbb{H}^2 - \{w\}$. Using $< .,. >$ to identify the cotangent bundle of $\mathbb{H}^2$ with the tangent bundle we can interpret this form as a vector field on $\mathbb{H}^2 - \{w\}$. Its value at x is the unit tangent vector at x along the geodesic running from w to x.

Let x_t and y_t be differentiable curves in $\mathbb{H}^2$ such that $x_t \neq y_t$.

$$\frac{d}{dt} d(x_t, y_t) \;=\; < x_t', \gamma > - < y_t', \gamma >$$

where γ is the geodesic running from y_t to x_t. (This is a special case of the First Variation Theorem.)

By taking x_t and y_t to be integral curves of a smooth vector field X on $\mathbb{H}^2$ we deduce the following.

The vector field X is a Killing field if and only if for each geodesic γ the component of X along γ is constant over γ.

4.3 The approximating diffeomorphism

The following notation is fixed for the whole of this section. Let $\mathbb{D}_1$ and $\mathbb{D}_2$ be two copies of $\mathbb{H}^2$. Let $E : \mathbb{D}_1 \to \mathbb{D}_2$ be a uniformly bounded left earthquake with source lamination λ and shearing metric μ. Let $E_t : \mathbb{D}_1 \to \mathbb{D}_2$ for $t \in [0, 1]$ be a smooth 1-parameter family of left earthquakes such that E_t has shearing metric $t\mu$ and $E_1 = E$. For each $x \in \mathbb{D}_2$ let $W_{x,t}$ and $V_{x,t}$ be the Killing fields on $\mathbb{D}_2$ defined in Section 4.1.

Let V_t be the vector field on $\mathbb{D}_2$ defined by $V_t(x) = V_{x,t}(x)$. Let $\Psi_t : \mathbb{D}_1 \to \mathbb{D}_2$ be the smooth 1-parameter family of diffeomorphisms which solves the equation $\Psi_t' = V_t$ subject to the initial condition that $\Psi_0 = E_0$. We call Ψ_t the *approximating diffeomorphism for* E_t.

We prove next that our construction of Ψ_t is in some sense natural. Let H_t be a smooth 1-parameter family of isometries of $\mathbb{D}_2$. Let $F_t = H_t \circ E_t$. Let

Ψ_t and Φ_t be the approximating diffeomorphisms for E_t and F_t respectively.

Lemma 4.4 *With the above notation we have* $\Phi_t = H_t \circ \Psi_t$ *for* $t \in [0,1]$.

Proof: We have $(F_t|z) = H_t \circ (E_t|z)$. Differentiating we obtain

$$(F_t|z)' = H_t' + H_t.(E_t|z)'.$$

Let $U_{y,t}$ be the approximating vector field for F_t.

$$\begin{aligned} U_{y,t} &= \int_{\mathbb{D}_1} (\phi_y \circ F_t)(z)(F_t|z)' dA(z) \\ &= \int_{\mathbb{D}_1} (\phi_{H_t^{-1}y} \circ E_t)(z)(H_t' + H_t.(E_t|z)')dA(z) \\ &= H_t' + H_t.V_{H_t^{-1}y,t} \end{aligned}$$

where $V_{y,t}$ is the approximating vector field for E_t. Let U_t be the vector field taking the value $U_{y,t}(y)$ at y and define V_t similarly. We have

$$\begin{aligned} \Phi_t' &= U_t \\ &= H_t' + H_t.V_t \\ &= (H_t \circ \Psi_t)'. \end{aligned}$$

Since $H_t \circ \Psi_t$ also satisfies the initial condition $H_0 \circ \Psi_0 = F_0$ the lemma follows from the uniqeness of solutions of ordinary differential equations. □

Let E_t be as before and let Γ be a group of isometries of $\mathbb{D}_1$ which leave the source lamination and shearing metric of E_t invariant. From Section 2.5 we know that each $g \in \Gamma$ induces an isometry h_t of $\mathbb{D}_2$ where

$$h_t \circ E_t = E_t \circ g.$$

Let Ψ_t be the approximating diffeomorphism for E_t. The approximating diffeomorphism for the right-hand side of the above equation is simply $\Psi_t \circ g$. For the left-hand side, by Lemma 4.4, we obtain $h_t \circ \Psi_t$. Therefore

$$h_t \circ \Psi_t = \Psi_t \circ g.$$

It follows that if E is an earthquake between surfaces then Ψ_1 gives a diffeomorphism between the same two surfaces.

Lemma 4.5 *Let* E_t *and* Ψ_t *be as before. The diffeomorphism* Ψ_1 *is quasi-isometric.*

Proof: Let z and y be distinct points in $\mathbb{D}_1$. Let x_t and y_t be the paths $\Psi_t(x)$ and $\Psi_t(y)$ respectively for $t \in [0,1]$. Let γ_t be the geodesic joining x_t to y_t. As at the end of Section 4.2 we see that

$$\frac{d}{dt}d(x_t, y_t) = < (V_{x_t,t} - V_{y_t,t})(x_t), \gamma >$$

and this implies

$$\left|\frac{d}{dt}d(x_t, y_t)\right| \leq \|(V_{x_t,t} - V_{y_t,t})(x_t)\|.$$

From the definitions we have

$$\begin{aligned}(V_{x_t,t} - V_{y_t,t})(x_t) &= \int_{\mathbb{D}_1}(\phi_{x_t} - \phi_{y_t})(z)W_{z,t}(x_t)dA(z) \\ &= \int_{\mathbb{D}_1}(\phi_{x_t} - \phi_{y_t})(z)(W_{z,t} - W_{x_t,t})(x_t)dA(z).\end{aligned}$$

For $z \in \operatorname{supp}(\phi_{x_t} - \phi_{y_t})$ clearly $d(z, x_t) \leq \delta + d(x_t, y_t)$. Suppose that M and η are the constants of uniform boundedness on μ_t and that $\delta + d(x_t, y_t) \leq \eta$. Then for all $z \in \operatorname{supp}(\phi_{x_t} - \phi_{y_t})$ Inequality 5 implies

$$\|(W_{z,t} - W_{x_t,t})(x_t)\| \leq M \cosh^2 \eta.$$

Let

$$k = 2M \cosh^2 \eta . \sup \|d\phi_{x_t}\| .(\text{area of a hyperbolic disk radius } \delta).$$

Whenever $\delta + d(x_t, y_t) \leq \eta$ we have

$$\left|\frac{d}{dt}d(x_t, y_t)\right| \leq k d(x_t, y_t). \tag{6}$$

Suppose finally that $\delta \leq \eta/2$. It is easy to deduce from Inequality 6 that, whenever $d(x_0, y_0) \leq e^{-k}\eta/2$,

$$d(x_1, y_1) \leq e^k d(x_0, y_0).$$

Clearly the condition $d(x_0, y_0) \leq e^{-k}\eta/2$ is redundant. The inequality can be rewritten

$$d(\Psi(x), \Psi(y)) \leq e^k d(x, y).$$

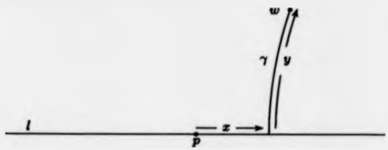
Similar reasoning gives the same inequality for Ψ^{-1}. Therefore Ψ is quasi-isometric. □

Lemma 4.6 *Let E_t and Ψ_t be as before. For each $\epsilon > 0$ there exists $\delta > 0$ (used in the definition of Ψ_t) such that the graph of Ψ_1 is ϵ-dense in the graph of E_1. (We use the supremum metric on $\mathbb{D}_1 \times \mathbb{D}_2$.)*

Proof: Let λ and μ be the source lamination and shearing metric respectively of E_1. Let p be a point in $\mathbb{D}_1$. We show that there is a $q \in \mathbb{D}_1$ such that both $d(p,q)$ and $d(E_1(p), \Psi_1(q))$ are $O(\delta)$. The proof is divided into 3 cases.

Case 1: *p lies on a leaf l of λ.* In view of Lemma 4.4 we can assume that $(E_t|l)$ is fixed. We use the same letters l and p for the images under E_t of l and p.

Define a coordinate system on $\mathbb{D}_2$ as follows. Fix an orientation of l so that the words *right of p* are meaningful. Let w be any point of $\mathbb{D}_2$. Let γ be the unique geodesic perpendicular to l through w. We define $x(w)$ by saying that γ crosses l a distance $x(w)$ to the right of p. We define $y(w)$ to be the (signed) distance of w from l.



Fix the following notation for the remainder of the proof of Case 1. Let q be a point in $\mathbb{D}_1$. Let $q_t = \Psi_t(q)$ and let γ_t be the complete geodesic perpendicular to l through q_t. We proceed by finding bounds on $dx(q_t')$ and $dy(q_t')$.

Suppose that

1. $y(q_t) + \delta \leq \sinh^{-1}(1)$ and that
2. $y(q_t) - \delta > 0$.

Let s be any point of the δ-ball centered at q_t. From 1 we deduce that any geodesic which weakly separates s from l must cross γ_t. In particular the axis of the vector field $W_{s,t}$ crosses γ_t. From 2 we see that s lies above l. Therefore the axis of $W_{s,t}$ crosses γ_t above l. Since E_t is a *left* earthquake, we deduce that $dx(W_{s,t}(q_t)) \leq 0$. Following through the definitions we see that this implies $dx(q_t') \leq 0$.

If instead we suppose that

1. $-y(q_t) + \delta \leq \sinh^{-1}(1)$ and that
2. $-y(q_t) - \delta > 0$

then we obtain $dx(q_t') \geq 0$. We have now established the bounds we will need on $dx(q_t')$.

We now consider $dy(q_t')$. Suppose that

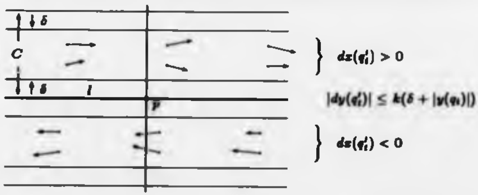
$$y(q_t) + \delta \leq \min\{\sinh^{-1}(1), \eta\}$$

where η, M are constants of uniform boundedness for E_q. Inequality 5 implies that the magnitude of $W_{s,t}$ on its axis is at most $M \cosh^2 \eta$. The cosine of the angle between γ_t and the axis of $W_{s,t}$ is at most $\sinh(\delta + |y(q_t)|)$. We have

$$\begin{aligned} |dy(W_{s,t}(q_t))| &= |< W_{s,t}, \gamma >| \\ &\leq M \cosh^2(\eta) \sinh(\delta + |y(q_t)|) \\ &\leq \left(\frac{M \cosh^2(\eta)}{\sinh^{-1}(1)} \right) (\delta + |y(q_t)|). \end{aligned}$$

Let $k = M \cosh^2(\eta)/\sinh^{-1}(1)$. Following through the definitions we obtain $|dy(q_t')| \leq k(\delta + |y(q_t)|)$. This is the bound we will need on $dy(q_t')$. Let $C = \min\{\sinh^{-1}(1), \eta\}$.

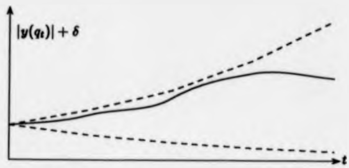
The diagram below summarises the inequalities we have obtained so far.



We have shown that

$$\delta + |y(q_t)| \le C \quad \Rightarrow \quad |dy(q_t')| \le k(\delta + |y(q_t)|).$$

Therefore, for $y(q_t)$ within the given range of values, the graph of $|y(q_t)| + \delta$ is constrained to lie between two exponential curves.



It follows that if $|y(q_0)| + \delta \le Ce^{-k}$ then

$$|y(q_t)| + \delta \le C$$

for all $t \in [0,1]$. Suppose now that $Ce^{-k} \ge (y(q_0) + \delta) \ge 2\delta e^{k}$ and that $x(q_0) = 0$. (We assume that $\delta \le Ce^{-2k}/2$.) It follows that

$$C \ge (y(q_t) + \delta) \ge 2\delta$$

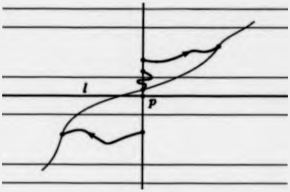
for all $t \in [0,1]$. From our inequalities concerning $dx(q_t')$ we deduce that

$$x(q_1) \le 0.$$

If instead we suppose that $Ce^{-k} \ge (-y(q_0) + \delta) \ge 2\delta e^{k}$ and $x(q_0) = 0$ then it follows that

$$x(q_1) \ge 0.$$

Since $x(q_1)$ depends continuously on q_0 we can find q_0 satisfying $x(q_0) = 0$ and $|y(q_0)| + \delta \le 2\delta e^{k}$ such that $x(q_1) = 0$.



For the same q_0 we see that $|y(q_1)| + \delta \leq 2\delta e^{2h}$. Recall that p is the point with coordinates $x(p) = y(p) = 0$. To summarise we have found a point $q = E_0^{-1}(q_0)$ such that $d(p,q) \leq (2e^h - 1)\delta$ and $d(E_1(p), \Psi_1(q)) \leq (2e^{2h} - 1)\delta$.

Case 2: *p lies within distance δ of a leaf l on $\partial\hat{p}$.* Let p' be a point on l such that $d(p,p') \leq \delta$. Clearly changing E_t by leaf variance does not change Ψ_t so we assume that $(E|l) = (E|p)$. Then $d(E_1(p), E_1(p')) \leq \delta$. By our previous results we can now find q such that $d(p,q) \leq 2e^h\delta$ and $d(E_1(p), \Psi_1(q)) \leq 2e^{2h}\delta$.

Case 3: *p does not lie within distance δ of any leaf in λ.* Then $E_t(p) = \Psi_t(p)$ for $t \in [0,1]$. This completes the proof of Lemma 4.6. □

Let E_t and Ψ_t be as defined at the start of this section. In view of Lemma 4.6 we suppose that for some $\epsilon > 0$ the graph of Ψ_t is ϵ-dense in the graph of E. We show that E and Ψ_1 give rise to the same map of ∂D_1 onto ∂D_2.

Let x_n be a sequence of points converging to a point $x \in \partial D_1$. The sequence $E(x_n)$ converges to $E(x)$ on ∂D_2. Let y_n be a sequence of points in D_1 such that $d(x_n, y_n) < \epsilon$ and $d(E(x_n), \Psi_1(y_n)) < \epsilon$. It follows that y_n converges to x and $\Psi_1(y_n)$ converges to $E(x)$. Therefore $E(x) = \Psi_1(x)$ as required.

We summarise the results of this section in our final theorem.

Theorem 4.7 *Let E be a uniformly bounded left earthquake between hyperbolic surfaces F_1 and F_2. There exists a quasi-isometric diffeomorphism $\Psi : F_1 \to F_2$ with the following property. The lifts of E and Ψ, to maps between the universal covers of F_1 and F_2, agree on the circle at infinity. Moreover the constant of quasi-isometry for Ψ depends only on the constants of uniform boundedness for E.*

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AUTHOR Oliver A. Goodman

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